High Dimensional Sphere Packing with Random Geometric Graphs

MMath Thesis

Jacob Green

University of Bath - Mathematical Sciencies

Preface

Abstract

Mirroring the work of Campos, Jennsen, Michelen and Sahasrabudhe in [3], we explore the intimate relationship between packings on unit-volume balls in $\mathbb{R}^d$ and the independence number of geometric graphs in $\mathbb{R}^d$. By sampling Poisson point processes in $\mathbb{R}^d$ at appropriate intensities and bounding the independence number of the induced geometric graph with radius twice that of the unit-volume ball in $\mathbb{R}^d$, we exhibit the existence of sphere packings density $\Theta(2^{-d}), \Theta(2^{-d}d)$ and $\Theta(2^{-d}d \log d)$ in $\mathbb{R}^d$. Our work is primarily concerned with the limiting regime $d \to \infty$, but bounds for fixed, large, d are given.

Disclaimer

We frequently write "it is clear that ..." when asserting upper and lower bounds, even though the omitted justifications may, on occasion, require nontrivial effort. This is particularly the case when the bounds have been shown by other authors. This shorthand keeps the exposition concise and focused. Each such bound has been verified with numerical experiments and traditional pen & paper derivations.

AI Statement

AI tools were used to correct minor LaTeX formatting errors and to conduct targeted literature searches. They were also used, on occasion, to write python scripts for numerical verification of bounds. Attempts were made to use AI tools for parts of proofs in Section 4, though the outcome was pitiful.

Contents

1 Introduction

1.1 Problem Statement

This thesis is concerned with the following meta problem.

Problem 1.1. What is the maximal density of d-dimensional Euclidean space we may cover with non-intersecting balls of unit volume?

Before formalising this, we introduce some notation that will be used throughout.

Equip $\mathbb{R}^d$ with the usual (Euclidean) norm, induced topology and measurable sets given by the Borel σ-algebra. Let $\mathrm{Vol}_d(A)$ denote[1] the Lebesgue measure in $\mathbb{R}^d$ of a measurable set $A \subset \mathbb{R}^d$ and let $\#B$ denote the counting measure of discrete (i.e. at most countable) B. Denote the ball centred at $x \in \mathbb{R}^d$ with radius $r > 0$ by $B(x; r) \subset \mathbb{R}^d$. When $\mathrm{Vol}(B(x; r)) = 1$ we call $B(x; r)$ a *d-ball* and we denote the radius of such balls by r_d.

For measurable $A \subseteq \mathbb{R}^d$ and a family of measurable sets $\mathcal{A}$ in $\mathbb{R}^d$, denote the *relative density* of $\mathcal{A}$ in A by

$$\mathcal{D}(\mathcal{A}|A) := \frac{1}{\mathrm{Vol}(A)} \mathrm{Vol}\left(\bigcup_{B \in \mathcal{A}} B \cap A \right)$$

It is clear $0 \leq \mathcal{D}(\mathcal{A}|A) \leq 1$ and that $\mathcal{D}(\mathcal{A}|A) \leq \mathcal{D}(\mathcal{A}|A')$ whenever $A \subseteq A'$ (i.e. $\mathcal{D}$ is monotone in its second argument).

We call a family of d-balls $\mathcal{P}$ a *sphere packing* of measurable $A \subseteq \mathbb{R}^d$ if the centres are in A and the balls have mutually disjoint interiors. We define the *sphere packing density* in $\mathbb{R}^d$ by

$$\theta(d) := \sup_{\mathcal{P}} \limsup_{R \to \infty} \mathcal{D}(\mathcal{P}|B(\mathbf{0}; R)) \tag{1}$$

where the supremum is taken over all sphere packings of $\mathbb{R}^d$. For sphere packings $\mathcal{P}$ of measurable $A \subseteq \mathbb{R}^d$ it is clear we have

$$\mathcal{D}(\mathcal{P}|A) \leq \frac{\#\mathcal{P}}{\mathrm{Vol}(A)} \tag{2}$$

The leeway in this inequality comes from the volume of the d-balls outside of A.

Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body (i.e. a convex set with nonempty interior). We assume all convex bodies have finite volume. Define the *L-dilation*[2] of $\mathcal{C}$ as the set $L\mathcal{C} := \{Lc : c \in \mathcal{C}\}$.

[1] When d is clear from context we may drop the subscript and simply write $\mathrm{Vol}(A)$.

[2] These are often called *hometheties* and if a convex body $\mathcal{C}' = L\mathcal{C}$ for some $L \in \mathbb{R}$ we say $\mathcal{C}'$ and $\mathcal{C}$ are *homothetic*.

It is clear for $L > 0$ that $L\mathcal{C}$ is also a convex body. It is known that convex sets in $\mathbb{R}^d$ are measurable, however the proof is nontrivial [20]. In section 2.1 we'll prove that we may equivalently define $\theta(d)$ with dilations of any convex body. We will also see that the supremum and limit supremum in (2) commute. This commutativity is integral to our arguments in section 4 as it allows us to pack in a body of finite volume.

In this thesis, we are primarily concerned with lower bounds for $\theta(d)$ in the limiting regime $d \to \infty$. Before getting to this, we provide a brief history of the problem in lower dimensions and upper bounds for $\theta(d)$ in the limiting regime.

1.2 Sphere packing In Low Dimensional Space

For $d \in \{1, 2, 3, 8, 24\}$ the explicit value of $\theta(d)$ is known. The case $d = 1$ is trivial, we may pack $\mathbb{R} \setminus \mathbb{Z}$ with $\mathcal{P} = \{[i, i+1] : i \in \mathbb{Z}\}$ to see $\theta(1) = 1$. The case $d = 2$ already requires some ingenuity but is nonetheless classical. Thue was able to prove[3], among other things, $\theta(2) = \frac{\pi}{2\sqrt{3}}$ in 1892 and that this density is obtained by a hexagonal tiling of unit-area circles in the plane [24]. One of the cleaner ways to compute $\theta(2)$ is to see that the Voronoi cells for the hexagonal packing are minimal.

The case $d = 3$ is a classical conjecture of Keplar, who predicted $\theta(3) = \frac{\pi}{3\sqrt{2}}$. There is a natural guess for an optimal packing here, namely stacking hexagonal arrangements of spheres on top of eachother so that the layers are nestled as deeply as possible into the previous. The proof that this packing is indeed optimal was not given until 2005 and is a result of Hales that required significant computer assistance [11]. In 2015 this proof was verified at the level of formal logic [12].

The case $d = 8$ was a recent breakthrough of Viazovska. She was able to prove $\theta(8) = \frac{\pi^4}{384}$, where the densest packing is given by placing 8-balls with centres at points on the E_8 lattice [34]. Her idea was to exploit the symmetries of $\mathbb{R}^8$ to show that linear programming (upper) bounds for $\theta(8)$ are sharp and obtained by the E_8 lattice. Shortly after, Cohn was able to build upon Viazovska's work to show that $\theta(24) = \frac{\pi^{12}}{12!}$ and that the optimal sphere packing is given by placing 24-balls with centres given by points on the Leech lattice [8]. In part due to her work on sphere packings, Viazovska was awarded a Fields medal in 2022.

For other small values of d we have reasonably sharp bounds given by linear programming, but proving optimality remains generally out of reach [7]. For arbitrarily large d the situation becomes far more complex. We can no longer take a linear programming approach as this argument varies dimension to dimension. The best known upper bound in the for arbitrary d is due to Kabatiansky and Levenshtein, who were able to prove in 1978 that

[3] His proof was not entirely rigorous, the first full proof was due to Tóth over 50 years later. [32].

$\theta(d) \leq 2^{(-0.599\cdots+o(1))d}$ where $o(1) \to 0$ as $d \to \infty$ by using the zonal spherical functions associated with a motion groups of space [17]. We will call results regarding the case $d \to \infty$ results in the *limiting regime*.

1.3 Sphere Packing In The Limiting Regime

By using a simple greedy algorithm we can prove $\theta(d) \geq 2^{-d}$, which we call the *trivial bound*. Fix $R > 0$ and let $\mathcal{P}$ be a sphere packing of $B(\mathbf{0}; R)$ such that for all points $x \in \mathbb{R}^d$ there exists a d-ball $B \in \mathcal{P}$ with the intersection $B \cap B(x; r_d)$ having nonempty interior. We interpret this as a sphere packing where we may cannot add any further d-balls without altering the structure of $\mathcal{P}$ and we call such packings *saturated*. It is clear that $\cup_{B\in\mathcal{P}} 2B \supseteq B \cap B(x; r_d)$, so by the subadditivity of measure

$$1 \leq \frac{\mathrm{Vol}(\cup_{B\in\mathcal{P}}(2B \cap B(\mathbf{0}; R)))}{\mathrm{Vol}(B(\mathbf{0}; R))} \leq \frac{1}{\mathrm{Vol}(B(\mathbf{0}; R))} \sum_{B\in\mathcal{P}} 2^d \mathrm{Vol}(B \cap B(\mathbf{0}; R)) \tag{3}$$

The trivial bound follows by multiplying through by 2^{-d} and letting $R \to \infty$.

We are yet to find an exponential improvement on the lower bound $\theta(d) \geq 2^{-d}$ for large d. Further, a conjecture of Parisi and Zamponi[4] predicts that $\theta(d) \leq (1 + o(1))2^{-d} d \log d$ for all *amorphous*[5] sphere packings [25] [26]. We will come back to the Parisi-Zamponi conjecture shortly.

Non-trivial lower bounds for $\theta(d)$ in the limiting regime are generally given by non-constructive combinatorial arguments that exhibit the existence of a packing of density equal to the claimed lower bound. The first improvement on the so-called trivial bound is usually attributed to Minkowski[6], but was first formally proven by Hlawka, in a result now known as the Minkowski-Hlawka theorem that states $\theta(d) \geq 2^{-d+1}\zeta(d) = (1 + o(1))2^{-d+1}$ where ζ denotes the Reimann Zeta function and $o(1) \to 0$ as $d \to \infty$ [23] [13]. This bound asymptotically beats the trivial bound by a factor of 2.

Shortly after Hlawka's proof, in 1947, Rogers was able to sharpen the techniques used by Hlawka to obtain the bound $\theta(d) \geq (1 + o(1))cd2^{-d}$ where $c = \frac{2}{e}$ and again $o(1) \to 0$ as $d \to \infty$. For the next 70 years this constant c was sharpened, with many recent developments utilising the Siegel mean value theorem (a generalisation of the previously mentioned Minkowski-Hlawka theorem). Venkatesh was able to obtain an extra factor of $\log \log d$ in a sparse sequence of dimensions [33]. Henry Cohn gives a reasonably deep account of these

[4]This conjecture comes from the famed replica method, ubiquitous in statistical physics. While not mathematically rigorous, it has an uncanny ability to correctly predict things.

[5]This is a loosely defined term from statistical physics. We may interpret it as a sphere packing that exhibits no global structure, e.g. non-periodic and certainly non-lattice.

[6]Minkowski merely stated the result as a consequence of his work on the geometry of numbers.

developments in [6].

Each of the afforementioned lower bounds for $\theta(d)$ in the limiting regime are proven by exhibiting the existence of lattice in $\mathbb{R}^d$ that induces a packing by placing d-balls with centres at lattice points. We call such packings *lattice packings*. Intuitively, we do not expect lattice packings to be optimal as $d \to \infty$. Volume scales exponentially in d, whereas a lattice is specified with a polynomial number of parameters in d. Further, depending on the dimension, the behaviour sphere packings greatly varies. The problem in $\mathbb{R}^7$ deviates greatly from the problem in $\mathbb{R}^8$. We do not expect to be able to handle this exponentially growing vacuum of volume with polynomial degrees of freedom, though the tools to prove that this is the case are unbeknownst to us. In fact, one may reasonably conjecture that as $d \to \infty$ no lattice packing will even be saturated. This conjecture again remains out of reach with our current toolkit.

Recall the so-called amorphous packings mentioned in the Parisi-Zamponi conjecture. By our above dialogue, we expect such packings to be optimal in the limiting regime. Hence we may further conjecture that as $d \to \infty$ we have $\theta(d) \leq (1 + o(1))2^d d \log d$. Whether you believe the true upper bound for $\theta(d)$ is given by the Parisi-Zamponi conjecture or closer to the bound given by Kabatiansky and Levenshtein depends on your view of *phase transitions*[7] in high dimensions.

If we wish to deviate from lattice packings, we can make the following observation.

Observation 1.2. Let $\mathbf{X} \subset \mathbb{R}^d$ be a discrete collection of points. Consider the so-called *geometric graph* with G vertices $\mathbf{X}$ and radius $2r_d$. In G, edges are given by the relation $x \sim y \Leftrightarrow \|x - y\| \leq 2r_d \wedge x \neq y$. An independent set in G corresponds with a sphere packing in $\mathbb{R}^d$.

Now we can split our problem into two key components. The first is to choose a "good" set of points $\mathbf{X} \subset \mathbb{R}^d$ and the second is to prove the geometric graph $G(\mathbf{X}, 2r_d)$ with vertices $\mathbf{X}$ and radius $2r_d$ has a large independent set. We dedicate one section of this thesis to each of these problems, beginning in section 3 with the independent set problem (the discrete component) and finishing in section 4 with the "good points" problem (the geometric component). We will tackle both of these with stochastic methods, applying the probabilistic method in combinatorics [2].

Historically, Krivelevich and Vardy were the first to use this observation to obtain a non-trivial bound on $\theta(d)$. In 2004 they were able to show $\theta(d) \geq cd2^{-d}$ where $c = 0.01$ in their

[7] We could dedicate a whole thesis to the discussion of this problem, but unfortunately I must only give it this footnote. The problem is concerned with random sphere packings at a given density. As this density increases, we expect packings to transition from "gas-like" (i.e. amorphous) to "lattice-like", but whether this is true remains unclear even in dimension 3.

packing. using a well-known lower bound on the independence number due to Ajtai, Komlós and Szemerédi [1] [18]. We will prove a stronger version of this independence number bound and this sphere packing bound in sections 3.2 and 4.2[8] respectively.

Of particular interest is the fact that their methods give rise to a deterministic procedure for describing the d-balls in their packing with complexity $O(2^{\gamma d \log_2(d)})$ for an absolute constant γ. Any deterministic procedure using observation 1.2 will have to have exponential complexity in d as it is well known finding large independent sets in graphs is NP, so unless $P = NP$ we're forced into infeasible algorithms in the limiting regime [35] [16].

On the December 15th 2023, mere months before the writing for this thesis began, Campos, Jennsen, Michelen and Sahasrabudhe released a preprint showing in the limiting regime we have $\theta(d) \geq (1 - o(1))2^{-d-1} d \log d$ where as usual $o(1) \to 0$ as $d \to \infty$ [3]. This was the first non-constant improvement since the work of Roger's almost 70 years prior and is regarded as a monumental breakthrough in high-dimensional sphere-packing.

By randomly sampling $\mathbf{X}$ according to a point process and bounding the independence number $\alpha(G)$ for the geometric graph G with vertex set $\mathbf{X}$ and radius $2r_d$, we obtain proofs that $\theta(d) = \omega(2^{-d}), \omega(2^{-d}d), \omega(2^{-d} d \log d$. We prove $\theta(d) = \omega(2^{-d})$ by deleting points of high degree from G and using the simple bound $\alpha(G) \geq \#V(G)/(\Delta(G)+1)$. By further deleting a vertex from each triangle in G, we can use Shearer's theorem to prove $\theta(d) = \Omega(2^{-d}d)$. The breakthrough idea of Campos, Jennsen, Michelen and Sahasrabudhe is to instead delete points of high codegree and to use "Shearer-like" theorem to bound $\alpha(G)$. This allows to sample $\mathbf{X}$ at a far greater intensity, giving the extra $\log d$ factor.

We note that the bound of Campos, Jennsen, Michelen and Sahasrabudhe is a factor of 2 off proving the Parisi-Zamponi conjecture. In [3] it is predicted that their independence number bound can be improved by a factor of 2, which would obtain one direction of the Parisi-Zamponi conjecture as random packings are amorphous. This missing factor of 2 has deep ties with Ramsey Theory. It is believed Shearer's theorem can be improved by a factor of 2 and that this factor of 2 is one of the two factors of 2 seperating upper and lower bounds of the off-diagonal Ramsey number $R(3, k)$ [28]. The key bridge here is the triangle-free process and its application to bounding $R(3, k)$, an idea originally attributed to Erdös and Bollobás[9].

[8] It is also worth mentioning I provided a sketch proof that $\theta(d) = \Omega(2^{-d}d)$ prior to my knowledge of [3] and [18], though our methods are eerily similar. Alas, I was too late to get an interesting contribution.

[9] Supposedly this was first discussed at the conference "Quo Vadis, Graph Theory?" at the University of Alaska [5].

1.4 Notation

We will use the following asymptotic notation conventions throughout. Suppose $f, g : \mathbb{R}_+ \to \mathbb{R}_+$. Then,

- $f = O(g)$ if $\exists c > 0$ such that for all x sufficiently large $f(x) \leq cg(x)$.
- $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if both $f = O(g)$ and $f = \Omega(g)$
- $f = o(g)$ if $f(x)/g(x) \to 0$ as $x \to \infty$ and $f \sim g$ if $f = (1 + o(1))g$.

We will use # to denote the counting measure of a countable set, whereas $\mathrm{Vol}_d(\cdot)$ will denote the Lebesgue measure in $\mathbb{R}^d$. When it is clear what dimension is being worked in we may drop the subscript. Throughout we equip $\mathbb{R}^d$ with the usual (Euclidean) norm, denoted $\|\cdot\|$, the usual topology induced by this norm and measurable sets given by the Borel σ-algebra.

Denote the radius of a d-ball with unit-volume by r_d, that is let r_d be the unique solution to $B(\mathbf{0}, r) = 1$, and let $B(x; r)$ denote the Euclidean ball centred at x with radius r. Let S^{d-1} denote the boundary of the d-dimensional ball $B(\mathbf{0}; 1)$ (the d-sphere).

For discrete $\mathbf{X} \subset \mathbb{R}^d$ let $G(\mathbf{X}, r)$ denote the geometric graph with vertex set $\mathbf{X}$ and radius r. The edges of this graph are given by the relation $x \sim y \Leftrightarrow x \neq y \wedge \|x - y\| \leq r$.

All logs, unless otherwise specified, are base e. Throughout $\overline{A}$ denotes the closure of $A \subset \mathbb{R}^d$, e.g. $\overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$ and ∂A denotes the boundary of A, e.g. $\partial[0, 1] = \{0, 1\}$. We denote the positive reals with $\mathbb{R}_+$ and the nonnegative integers with $\mathbb{N}_0$.

Throughout, for $x \in \mathbb{R}_+$, $\Gamma(x)$ will denote the classical Gamma function[10]. Namely,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t \tag{4}$$

It is well known (and easy to prove) that this function obeys the recursion $\Gamma(x+1) = x\Gamma(x)$ and hence as $\Gamma(1) = 1$, we have $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}_0$.

See section 1.1 for notation pertaining to $\theta(d)$, e.g. relative density and $\theta(d)$ itself. See sections 2.3 and 3 for notation pertaining to non-geometric graph theory.

[10] We could equally define this for $z \in \mathbb{C}$ with $\Re(z) > 0$, but everything in this thesis is real.

2 Preliminaries

We split this section into three subsections.

2.1 Geometry: Formulas for $\mathrm{Vol}(B(\mathbf{0},r))$, $\mathrm{Vol}(\partial B(\mathbf{0},r))$ and r_d, bounds and asymptotics for r_d and proofs for the claims made in section 1.1.

2.2 Poisson Processes: Concentration inequalities for Poisson random variables, definitions and basic facts regarding the Poisson point process and Mecke's Equation.

2.3 Graph Theory & The Probabilistic Method: Basic graph theoretic notions, Turán's theorem, two basic lemmas from probability theory, proof graphs have large bipartite subgraphs and a lower bound on the diagonal Ramsey number.

2.1 Geometry

We begin with formulae for the volume of the ball $B(\mathbf{0},r)$, the volume of its boundary and the radius of the d-ball r_d. Before doing so, a quick technical lemma.

Lemma 2.1. For $n \in \mathbb{N}_0$,

$$W_n := \int_0^{\frac{\pi}{2}} (\sin\theta)^n \,\mathrm{d}\theta$$

Then, for $m \in \mathbb{N}_0$, we have

$$W_{2m} = \frac{\pi}{2}\cdot\frac{\Gamma(2m+1)}{2^{2m}\Gamma(m+1)^2} \quad \text{and} \quad W_{2m+1} = \frac{2^{2m}\Gamma(m+1)^2}{\Gamma(2m+2)}$$

Proof. By applying the Pythagorean identity $(\sin\theta)^2 + (\cos\theta)^2 = 1$, recurse

$$W_n = W_{n-2} - \int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2} (\cos\theta)^2 \,\mathrm{d}\theta$$

This remaining integral is handled by parts[11]. We compute

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2} (\cos\theta)^2 \,\mathrm{d}\theta = \left[\frac{(\sin\theta)^{d-1}}{d-1}\cos\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} + \frac{1}{n-1}W_n = \frac{1}{n-1}W_n$$

Hence,

$$W_n = \frac{n-1}{n}W_{n-2}$$

Computing W_0, $W_1 = \frac{\pi}{2}$, 1 and solving the recursion finishes the proof. □

Remark 2.2. This integral is well-known and is usually called the *Wallis integral.*

[11] We take $\mathrm{d}v = \cos\theta\,(\sin\theta)^{n-2}$ and $u = \cos\theta$.

Lemma 2.3. Let $r > 0$ and $d \in \mathbb{N}$. Then, in $\mathbb{R}^d$,

$$\mathrm{Vol}_d(B(\mathbf{0}, r)) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^d \quad \text{and} \quad \mathrm{Vol}_{d-1}(\partial B(\mathbf{0}, r)) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} r^{d-1}$$

Furthermore,

$$r_d = \left(\frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{\frac{d}{2}}}\right)^{\frac{1}{d}}$$

Idea: See that $\mathrm{Vol}(B(\mathbf{0}, r)) \propto r^d$ and use Fubini's theorem to obtain a recursion for $\mathrm{Vol}(B(\mathbf{0}, 1))$ between dimensions. After, simple calculus gives the formula for $\mathrm{Vol}(\partial B(\mathbf{0}, r))$.

Proof. Let V_d denote the volume of the d-dimensional ball $B(\mathbf{0}, 1)$. Observe

$$\mathrm{Vol}(B(\mathbf{0}, r)) = \int_{\|x\| \le r} \mathrm{d}x = r^d \int_{\|x\| \le 1} \mathrm{d}x = r^d V_d \tag{5}$$

where the penultimate equality follows from making the substitution $x \mapsto x/r$. Hence, to find $\mathrm{Vol}(B(\mathbf{0}, r))$, it suffices to find V_d. By Fubini's theorem

$$V_d = \int_{x_1^2+\cdots+x_d^2 \le 1} \mathrm{d}x = \int_{x_1^2 \le 1} \left(\int_{x_2^2+\cdots+x_d^2 \le 1-x_1^2} \mathrm{d}x_2 \cdots \mathrm{d}x_d\right) \mathrm{d}x_1 \tag{6}$$

By using equation (5) on the integrand the inside integral of (6), deduce

$$V_d = \left(\int_{-1}^{1} (1-x^2)^{d-1} \mathrm{d}x\right) V_{d-1}$$

Now we compute this integral. Make the substitution $x = \cos\theta$ to see

$$\int_{-1}^{1} (1-x^2)^{d-1} \mathrm{d}x = \int_0^{\pi} (\sin\theta)^d \, \mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^d \, \mathrm{d}\theta = 2W_d$$

where the penultimate equality follows from the symmetry of sin about $\frac{\pi}{2}$ and W_d denotes the Wallis integral. All together, we thus have

$$V_d = 2W_d \cdot V_{d-1} \xRightarrow{\text{Induct}} V_d = 2^{d-1} W_d W_{d-1} \cdots W_2 V_1 = 2^d W_d W_{d-1} \cdots W_2 \tag{7}$$

By lemma 2.1, we compute for $m \in \mathbb{N}_0$

$$W_{2m} W_{2m+1} = \frac{\pi}{2} \cdot \frac{\Gamma(2m+1)}{2^{2m}\Gamma(m+1)^2} \cdot \frac{2^{2m}\Gamma(m+1)^2}{\Gamma(2m+2)} = \frac{\pi/2}{2m+1} \tag{8}$$

Hence, using (7) and (8), we compute

$$\begin{aligned}
V_{2m} &= 2^{2m} W_{2m}(W_{2m-1}W_{2m-2})\cdots(W_3W_2) \\
&= 2^{2m} \cdot \frac{\pi}{2} \cdot \frac{\Gamma(2m+1)}{2^{2m}\Gamma(m+1)^2} \cdot \left(\prod_{i=0}^{m-2} \frac{\pi/2}{2m-(2i+1)}\right) \\
&= \frac{\pi\Gamma(2m+1)}{2\Gamma(m+1)^2} \left(\prod_{i=0}^{m-2} \frac{\pi(m-i)}{(2m-2i)(2m-(2i+1))}\right) \\
&= \frac{\pi\Gamma(2m+1)}{2\Gamma(m+1)^2} \cdot \frac{2\pi^{m-1}\Gamma(m+1)}{\Gamma(2m+1)} = \frac{\pi^m}{\Gamma(m+1)}
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
V_{2m+1} &= 2^{2m+1}(W_{2m+1}W_{2m})\cdots(W_3W_2) \\
&= 2^{2m+1} \prod_{i=0}^{m-1} \frac{\pi/2}{2m-(2i-1)} \\
&= \frac{\pi^m 2^{m+1}}{(2m+1)(2m-1)(2m-3)\cdots 3} \\
&= \frac{\pi^{m+\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}+1\right)}
\end{aligned}$$

where the final equality follows from $\Gamma(1/2) = \sqrt{\pi}$. Hence,

$$V_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \quad \xrightarrow{\text{equation (5)}} \quad \mathrm{Vol}(B(\mathbf{0},r)) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^d$$

Now for the volume of $\partial B(\mathbf{0}, r)$, apply the fundamental theorem of calculus to see

$$\mathrm{Vol}(B(\mathbf{0},r)) = \int_0^r \mathrm{Vol}(\partial B(\mathbf{0},r)) \xRightarrow{\text{FTC}} \mathrm{Vol}(\partial B(\mathbf{0},r)) = \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{Vol}(B(\mathbf{0},r)) \tag{9}$$

From (9), it is easy to deduce

$$\mathrm{Vol}(\partial B(\mathbf{0},r)) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} r^{d-1}$$

For r_d we solve $\mathrm{Vol}(B(\mathbf{0},r)) = 1$, which is trivial and completes the proof. □

It turns out this quantity $r_d \approx c\sqrt{d}$. The next lemma formalises this and gives explicit upper and lower bounds. First we'll need to extend Stirling's formula to the Γ function.

Lemma 2.4. For all $x > 0$,

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x+\epsilon(x)}$$

where $\epsilon(x) \in [0, \frac{1}{12x}]$.

Proof. Consult Theorem 1 in [15]. $\square$

Lemma 2.5. For $d \geq 2$,

$$r_d = \frac{C(d)}{\sqrt{2\pi e}}\sqrt{d+2}$$

where

$$\left(\frac{\pi(d+2)}{\exp(2)}\right)^{\frac{1}{2d}} \leq C(d) \leq \left(\frac{\pi(d+2)}{\exp\left(2 - \frac{1}{3d+6}\right)}\right)^{\frac{1}{2d}} \tag{10}$$

In particular, $r_d \sim \frac{1}{\sqrt{2\pi e}}\sqrt{d}$ as $d \to \infty$.

Proof. By lemma 2.3

$$r_d = \left(\frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{\frac{d}{2}}}\right)^{\frac{1}{d}} = \frac{1}{\sqrt{\pi}}\left(\Gamma\left(\frac{d}{2}+1\right)\right)^{\frac{1}{d}}$$

Combining with lemma 2.4, compute

$$r_d = \frac{1}{\sqrt{\pi}}\left(\sqrt{2\pi}\left(\frac{d}{2}+1\right)^{\frac{d}{2}+\frac{1}{2}} e^{-\frac{d}{2}-1+\epsilon(d)}\right)^{\frac{1}{d}} \tag{11}$$

where $\epsilon(d) \in (0, \frac{1}{6d+12})$. Simplifying equation (11), obtain

$$r_d = \frac{\left(\sqrt{\pi(d+2)}e^{\epsilon(d)-1}\right)^{\frac{1}{d}}}{\sqrt{2\pi e}}\sqrt{d+2} = \frac{C(d)}{\sqrt{2\pi e}}\sqrt{d+2}$$

It is straightforward to see $C(d) \to 1$ as $d \to \infty$ so that $r_d \sim \frac{1}{\sqrt{2\pi e}}\sqrt{d}$. $\square$

Remark 2.6. Our upper bound is very tight. For $d = 10$ we have an error of $\approx 10^{-6}$.

We will need the following bound on the intersection of two balls radius $2r_d$.

Lemma 2.7. Let $t \geq 0, d \geq 4$ and $x, y \in \mathbb{R}^d$ be such that $\|x - y\| \geq t$. Then

$$\mathrm{Vol}(B(x; 2r_d) \cap B(x; 2r_d)) \leq 2^d e^{-t^2/4}$$

Idea: Cover the intersection with an appropriately sized ball centred at the midpoint of the line segment with endpoints x, y.

Proof. Let $z \in \mathbb{R}^d$. We first observe we can write the distance of z from $\frac{x+y}{2}$ as a function of its distance from the points x, y and the size of the line segment $x + y$ by expanding norms as inner products as follows.

$$\begin{aligned}
4\left\|z - \frac{x+y}{2}\right\|^2 &= 4\left\langle z - \frac{x+y}{2}, z - \frac{x+y}{2}\right\rangle \\
&= 4\langle z, z\rangle - 4\langle z, x+y\rangle + \langle x+y, x+y\rangle \\
&= 2\langle z-x, z-x\rangle + 2\langle z-y, z-y\rangle - \langle x-y, x-y\rangle \\
&= 2\|z-x\|^2 + 2\|z-y\|^2 - \|x-y\|^2
\end{aligned}$$

Hence, if $z \in B(x; 2r_d) \cap B(y; 2r_d)$, we have

$$4\left\|z - \frac{x+y}{2}\right\|^2 \leq 16r_d^2 - t^2$$

and it is clear we have the covering

$$B(x; 2r_d) \cap B(y; 2r_d) \subseteq B\left(\frac{x+y}{2} \ ; \ \sqrt{4r_d^2 - \frac{t^2}{4}}\right)$$

Using lemma 2.3, deduce

$$\mathrm{Vol}\left(B\left(\mathbf{0} \ ; \ \sqrt{4r_d^2 - \frac{t^2}{4}}\right)\right) = 2^d\left(1 - \frac{t^2}{16r_d^2}\right)^{\frac{d}{2}} \leq 2^d \exp\left(-\frac{t^2 d}{32r_d^2}\right)$$

where we use the inequality $1 - x \leq e^{-x}$ to deduce the final inequality. Using lemma 2.5 we see that for $d \geq 4$ we have $r_d \leq \sqrt{\frac{d}{8}}$, establishing the lemma. $\square$

We now move on to proving the claims made in section 1.1. Lemma 2.8 is novel, as is our proof of lemma 2.9[12].

Lemma 2.8. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body. Then

$$\theta(d) = \sup_{\mathcal{P}} \limsup_{L\to\infty} \mathcal{D}(\mathcal{P}|L\mathcal{C})$$

where the supremum is taken over sphere packings $\mathcal{P}$ of $\mathbb{R}^d$.

[12] At least, to the best of my knowledge. In [3] lemma 2.9 is stated for $\mathcal{C} = B(\mathbf{0}, 1)$. They claim standard arguments show it holds. I believe our arguments are standard and likely what they had in mind.

Proof. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body. For each $L > 0$ choose $R_L > 0$ so that $B(\mathbf{0}; R_L) \supseteq L\mathcal{C}$. Observe

$$\mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \mathcal{D}(\mathcal{P}|B(\mathbf{0}; R_L)) \leq \limsup_{R\to\infty} \mathcal{D}(\mathcal{P}|B(\mathbf{0}; R))$$

for all sphere packings $\mathcal{P}$ of $\mathbb{R}^d$. Taking the limit supremum as $L \to \infty$, see

$$\limsup_{L\to\infty} \mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \limsup_{R\to\infty} \mathcal{D}(\mathcal{P}|B(\mathbf{0}; R))$$

Taking the supremum over such $\mathcal{P}$, deduce

$$\sup_{\mathcal{P}} \limsup_{L\to\infty} \mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \sup_{\mathcal{P}} \limsup_{R\to\infty} \mathcal{D}(\mathcal{P}|B(\mathbf{0}; R)) = \theta(d)$$

For each $R > 0$ we seek $L_R > 0$ so that $L_R\mathcal{C} \supseteq B(\mathbf{0}; R)$. If we have such L_R we may carry out an analogous argument for the reverse inequality, completing the proof. To see that we can choose such L_R, it suffices to see that $\mathcal{C}$ contains a basis of $\mathbb{R}^d$. Choose a point c in the interior of $\mathcal{C}$. Then there exists $r > 0$ such that for all $1 \leq i \leq d$ we have $c + re_i \in \mathcal{C}$ where $\{e_1, \ldots, e_d\}$ is the standard basis of $\mathbb{R}^d$. □

Lemma 2.9. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body. Then

$$\theta(d) = \limsup_{L\to\infty} \sup_{\mathcal{P}} \mathcal{D}(\mathcal{P}|L\mathcal{C})$$

where the supremum is taken over sphere packings $\mathcal{P}$ of $L\mathcal{C}$. In particular, the supremum and limit surpremum in equation (1) commute.

Proof. For measurable $A \subseteq \mathbb{R}^d$, let $\mathrm{Pack}(A)$ denote the set of all sphere packings of A. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and let $L > 0$. Observe

$$\sup_{\mathcal{P}\in\mathrm{Pack}(L\mathcal{C})} \mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \sup_{\mathcal{P}\in\mathrm{Pack}(L\mathcal{C})} \limsup_{L\to\infty} \mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \sup_{\mathcal{P}\in\mathrm{Pack}(\mathbb{R}^d)} \limsup_{L\to\infty} \mathcal{D}(\mathcal{P}|L\mathcal{C})$$

as each sphere packing of $L\mathcal{C}$ is a sphere packing of $\mathbb{R}^d$. As the right hand side is independent of L, deduce

$$\limsup_{L\to\infty} \sup_{\mathcal{P}\in\mathrm{Pack}(L\mathcal{C})} \mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \theta(d)$$

For the reverse inequality, observe

$$\mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \sup_{\mathcal{P}\in\mathrm{Pack}(L\mathcal{C})} \mathcal{D}(\mathcal{P}|L\mathcal{C}) \implies \limsup_{L\to\infty} \mathcal{D}(\mathcal{P}|L\mathcal{C}) \leq \limsup_{L\to\infty} \sup_{\mathcal{P}\in\mathrm{Pack}(L\mathcal{C})} \mathcal{D}(\mathcal{P}|L\mathcal{C})$$

As the RHS is constant, we take supremums over $\mathrm{Pack}(\mathbb{R}^d)$ to deduce

$$\theta(d) \leq \limsup_{L\to\infty} \sup_{\mathcal{P}\in\mathrm{Pack}(L\mathcal{C})} \mathcal{D}(\mathcal{P}|L\mathcal{C})$$

which completes the proof. □

2.2 Poisson Processes

Throughout this subsection let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space and $\omega \in \Omega$. Let us begin with our concentration inequality for Poisson random variables[13].

Lemma 2.10. Let $\lambda > 0$ and let $Y \sim \mathrm{Po}(\lambda)$. Then for all $t \geq 0$

$$\mathbb{P}(Y - \lambda \geq t\lambda) \leq \exp(-\lambda \min\{t, t^2\}/3)$$

Idea: Mimic the proof of the Chernoff bound, evaluating the MGF near its minimum.

Proof. Let $a \geq 0$. Then, by applying Markov's inequality[14], for any $s > 0$

$$\mathbb{P}(Y - \lambda \geq t\lambda) = \mathbb{P}(e^{sY} \geq e^{s\lambda(t+1)}) \leq \mathbb{E}[e^{sY}]e^{-s\lambda(t+1)} = \exp\left(\lambda(e^s - 1) - s\lambda(t+1)\right)$$

Taking $s = \log(1 + t)$, we obtain

$$\mathbb{P}(Y - \lambda \geq t\lambda) \leq \exp\left(-\lambda((t+1)\log(t+1) - t)\right) \tag{12}$$

It suffices to bound $f(t) := (t+1)\log(t+1) - t$ below by $g(t) := \min\{t, t^2\}/3$. We do this with elementary calculus.

Case 1: $t \geq 1$
Consider $(f - g)(t) = (t+1)\log(t+1) - t - \frac{t}{3}$. Then $(f - g)(1) = 2\log(2) - \frac{4}{3} > 0$ and $(f - g)'(t) = \log(t+1) - \frac{1}{3} > 0$ for each $t \geq 1$. Hence we're increasing and bounded below by 0 and it is clear $g(t) \leq f(t)$ for $t \geq 1$.

Case 2: $0 \leq t < 1$
Consider $(f-g)(t) = (t+1)\log(t+1) - t - \frac{t^2}{3}$. By taylor expanding $\log(t+1) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots$ we see it is sufficient to prove $\frac{t^2}{2} - \frac{t^3}{6} \geq \frac{t^2}{3}$. This is easily seen by bounding $t^3 \leq t^2$.

Hence, replacing $f(t)$ with $g(t)$ in equation (12) completes the proof. □

We now move on the Poisson point process, the stochastic tool of choice in section 4. First, let us carefully define the *point process*. We will define this as a random measure but we will see that the Poisson point process can be defined as a collection of random points instead, matching our intuition for a point process.

[13] This lemma is stated in [3] without proof. Our proof is thus a contribution, though the argument is standard and has almost certainly been given elsewhere.

[14] Note we also use the well known moment generating function for the Poisson distribution, for which a derivation is elementary.

Definition 2.11. Let $\mathbf{N}_{<\infty}$ denote the space of $\mathbb{N}_0$-valued measures on $\mathbb{R}^d$ and let $\mathbf{N}$ denote the space of measures that can be written as countable sums in $\mathbf{N}_{<\infty}$. View $\mathbf{N}$ as a measure space equipped with measurable sets

$$\mathcal{N} := \sigma(\{\mu \in \mathbf{N} : \mu(B) = k\}) \qquad \text{for } k \in \mathbb{N}_0, B \in \mathcal{B}(\mathbb{R}^d) \tag{13}$$

A Point Process on $\mathbb{R}^d$ is a random element η of $(\mathbf{N}, \mathcal{N})$.

Let δ be the Dirac measure[15] on $\mathbb{R}^d$. If there are random points $X_1, X_2, \dots$ in $\mathbb{R}^d$ and an $\overline{\mathbb{N}_0}$-valued random variable κ such that

$$\eta = \sum_{n=1}^{\kappa} \delta_{X_n} \qquad \text{A.S.} \tag{14}$$

we call our point process η *proper*.

We call a measure *s-finite* if it can be expressed as a countable sum of finite measures. It is not too hard to see that every σ-finite measure is s-finite[16].

Definition 2.12. Let λ be an s-finite measure on $\mathbb{R}^d$. A *Poisson point process* with intensity measure λ is a point process η on $\mathbb{R}^d$ with the following two properties.

i. For measurable $B \subseteq \mathbb{R}^d$ the random variable $\eta(B)$ has $\mathrm{Po}(\lambda(B))$ distribution.

ii. For all $m \in \mathbb{N}$ and pairwise disjoint measurable $B_1, \dots, B_m$ the random variables $\eta(B_1), \dots, \eta(B_m)$ independent.

Note we define $X \sim \mathrm{Po}(\infty) \Rightarrow X = \infty$ A.S.

As one would hope, the Poisson point process for any given intensity measure exists. Further, it is proper. Hence we may identify a point process with a countable collection of random points $\mathbf{X} = (X_1, X_2, \dots)$ in $\mathbb{R}^d$. We define Poisson point processes on measurable $\Lambda \subseteq \mathbb{R}^d$ naturally by restriction. For such Λ and real-valued $\lambda > 0$ we will write $\mathbf{X} \sim \mathrm{PPP}_\Lambda(\lambda)$ for the collection of random points associated with a Poisson point process on Λ with intensity measure $\lambda \mathrm{Vol}(\cdot)$.

Poisson point processes obey the so-called *Mecke equation*, which allows for the rapid computation of expectations taken over a Poisson point process. In fact, this equation characterises the Poisson point process. We use a slightly less general result than that found in [21] which is both easier to prove and easier for us to apply. The version we state is a slight generalisation of the one found in [19]. We omit the proof as its analogous.

[15] That is, $\delta_x(A) = \begin{cases} 1 : x \in A \\ 0 : x \notin A \end{cases}$ for A measurable. Standard arguments show this is a measure.

[16] σ-finiteness gives a countable partition $(F_i)_{i \geq 1}$ of space such that the measure of each F_i is finite. Write our measure as the countable sum of restriction measures onto each F_i to deduce s-finiteness.

Theorem 2.13. Let $\Lambda \subset \mathbb{R}^d$ be bounded and measurable, let $\lambda > 0$ and let $\mathbf{X} \sim \mathrm{PPP}_\Lambda(\lambda)$. Let $k \in \mathbb{N}$ and let f be a real-valued measurable function defined on the product of $(\mathbb{R}^d)^k$ and the space of finite subsets of Λ. Then, when the expectation exists,

$$\mathbb{E}\left[\sum_{X_1,\ldots,X_k\in\mathbf{X}}^{\neq} f(X_1,\ldots,X_k,\mathbf{X}\setminus\{X_1,\ldots,X_k\})\right] = \lambda^k \int_\Lambda \cdots \int_\Lambda \mathbb{E}[f(x_1,\ldots,x_k,\mathbf{X})]\mathrm{d}x_1\cdots\mathrm{d}x_k$$

where $\sum^{\neq}$ denotes summation over k-tuples of distinct points $X_1 \neq \cdots \neq X_k$.

2.3 Graph Theory & The Probabilistic Method

Let $G = (V, E)$ be a graph. We assume all graphs are *simple*, i.e. have no loops and are finite. For vertices $u, v \in V$, we define the relation $u \sim v \Leftrightarrow \{u, v\} \in E$. We denote the *degree* of a vertex $v \in V$ by $d_G(v)$, which is equal to the number vertices adjacent to v (i.e. the size of the neighbourhood[17], $\mathcal{N}_G(v)$, of v). Let $\Delta(G) := \max_{v\in V} d_G(v)$ and $d(G) := \frac{1}{\#V}\sum_{v\in V} d_G(v)$ denote the maximal and average degrees of G respectively.

The *induced subgraph* with vertices $W \subseteq V$ is the graph constructed by deleting vertices in $V \setminus W$ and all edges incident[18] to these vertices. An independent set in G is a subset $I \subseteq V$ with the property that the induced subgraph with vertices I has no edges. We define the *independence number* of G the size of the largest independent set in G, and denote this quantity $\alpha(G)$. The goal of section 3 is to bound this quantity from below, which exhibits the existence of a large independent set in G. We'll begin with an easy one due to Turán.

Theorem 2.14. Let $G = (V, E)$ be a graph. Then

$$\alpha(G) \geq \frac{\#V}{\Delta(G) + 1}$$

Idea: Consider the obvious greedy algorithm for constructing an independent set.

Proof. Apply some ordering to V and let $I = \emptyset$. Choose the first element of V, v, add v to I and delete $\mathcal{N}_G(v) \cup \{v\}$ from V. Repeat until $V = \emptyset$. Each iteration of this algorithm deletes at most $\Delta(G) + 1$ vertices from V, hence we have at least $\frac{n}{\Delta(G)+1}$ iterations. It is clear that throughout the algorithm I remains an independent set. Thus we have an independent set of size at least $\frac{n}{\Delta(G)+1}$, establishing the lemma. $\square$

In section 3.1 we'll see that one can replace $\Delta(G)$ with the average degree $d(G)$ in our bound. Our proofs for further bounds on $\alpha(G)$ will all utilise the probabilistic method.

[17] In graph theory we use the convention that the neighbourhood of v, $\mathcal{N}_G(v)$, has $v \notin \mathcal{N}_G(v)$. This will be counterintuitive to the reader with a topology background, but in graph theory we often want to disclude v from our neighbourhood of v, hence the convention.

[18] That is, edges that have this vertex as one of its endpoints.

The idea of this method is to randomly sample from some set of constructions (e.g. graphs on $\{1, \dots, n\}$, subgraphs of K_n, colourings of $\mathbb{Z}$) and prove that our desired construction (e.g. a graph with a large bipartite subgraph, a long monochromatic progression) occurs with positive probability. This then affirms that the set of "desirable" constructions is nonempty, establishing existence.

We will only need two simple lemmas from probability to do this. One is just the (finite) subadditivity of measure and the other an easy homework problem. Combinatorialists[19] often refer to these as the union bound and first moment method respectively.

Lemma 2.15. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A_i \in \mathcal{F}, 1 \leq i \leq n$ be a finite family of events. Then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Lemma 2.16. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a real-valued random variable on this space. Then

$$\mathbb{P}\left(X \geq \mathbb{E}[X]\right) > 0 \quad \text{and} \quad \mathbb{P}\left(X \leq \mathbb{E}[X]\right) > 0$$

To get a feel for how to use these lemmas, we provide two quick expositions from graph theory. The first is a classical lemma that affirms every graph has a large bipartite subgraph, the second a lower bound on the Ramsey number due to Erdös in what is often regarded as the seminal paper on the probabilistic method[20] [9].

Proposition 2.17. Let $G = (V, E)$ be a graph with m edges. Then G has a bipartite subgraph H with at least $\frac{m}{2}$ edges.

Idea: Consider a random vertex colouring and show that $m/2$ of the edges of G have differently coloured endpoints via the first moment method.

Proof. Assign each vertex $v \in V$ a colour from $\{1, 2\}$ uniformly and independently at random. Let $E' \subseteq E$ be the set of edges that have differently coloured endpoints. Then, for $e \in E$, we have $\mathbb{P}(e \in E') = 1/2$ so that

$$\mathbb{E}[\#E'] = \mathbb{E}\left[\sum_{e \in E} \mathbf{1}\{e \in E'\}\right] = \sum_{e \in E} \mathbb{P}(e \in E') = \frac{m}{2}$$

by the linearity of expectations. Consider the subgraph H with vertex set V and edge set E'. From graph theory we know a graph is bipartite iff it is 2-colourable. It is clear H is 2-colourable, hence by lemma 2.16 there exists E' so that H is bipartite and $\#E' \geq \frac{m}{2}$. $\square$

[19] I include myself into the set of combinatorialists, though the University of Bath unfortunately has no such department.

[20] While his argument is phrased as a counting argument, the intuition is probabilistic. In general any application of the probabilistic method on finite spaces can be phrased as a counting argument, though for more involed arguments (e.g. those using the Lovaśz Local Lemma or Entropy) this is intractible.

The complete graph K_n on $\{1,\dots,n\}$ is the graph with $i \sim j$ for all $i \neq j$. We also refer to this as the *n-clique*. Define the *Ramsey number* $R(k,\ell)$ by

$$R(k,\ell) := \inf\{n \in \mathbb{N} : \forall \text{ 2-edge-colourings of } K_n \ \exists \text{ a monochromatic } k\text{-clique or } \ell\text{-clique}\}$$

That is, it is the smallest complete graph K_n such that no matter how we colour the edges of K_n with red/blue, we can find a red/blue complete subgraph K_k or K_ℓ. Ramsey's theorem (among other things) affirms that this quantity is finite [29][21]. We now bound the diagonal Ramsey number $R(k,k)$ from below.

Proposition 2.18. If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

Idea: Randomly colour the edges of K_n and show the probability of K_n containing a monochromatic K_k is less than 1.

Proof. Colour the edges of K_n with colours $\{1,2\}$ independently and uniformly at random. For k-subsets $S \subset V$ let A_S denote the event that the induced subgraph with vertex set S is monochromatic. It is clear that $\mathbb{P}(A_S) = 2^{1-\binom{k}{2}}$ and hence, letting[22] $\mathcal{S} = \binom{\{1,\dots,n\}}{k}$,

$$\mathbb{P}\left(\bigcap_{S\in\mathcal{S}} A_S^c\right) = 1 - \mathbb{P}\left(\bigcup_{S\in\mathcal{S}} A_S\right) \geq 1 - \sum_{S\in\mathcal{S}} \mathbb{P}(A_S) = 1 - \binom{n}{k}2^{1-\binom{k}{2}} \tag{15}$$

where the inequality is due to lemma 2.15. The intersection on the left hand side of (15) is simply the event that $R(k,k) > n$, which thus establishes the lemma. $\square$

Remark 2.19. By optimising n as a function of k, we obtain the quantitative bound

$$R(k,k) > \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{k/2}$$

This is only a factor of 2 behind the best known bound, found by Spencer in 1977 [31], which follows by using the Lovász local lemma to tighten the bound on $\mathbb{P}(\cup_{S\in\mathcal{S}} A_S)$.

[21] This theorem can be interpreted as "In all disordered systems there exists an ordered subsystem". Ramsey theory studies questions on this form on all sorts of objects, e.g. $\mathbb{Z}$, graphs and hypercubes.

[22] We use the common combinatorial notation $\binom{A}{k}$ here to denote the set of k-subsets of A.

3 Lower Bounds For $\alpha(G)$.

Recall the notation given in section 2.3. We define the codegree of distinct vertices $u, v \in V$ by $d_G(u, v) := \#\mathcal{N}_G(u) \cap \mathcal{N}_G(v)$. We define the maximal codegree of G, $\Delta_{\text{co}}(G)$, in the obvious way. This section is outlined as follows:

- Turán, Caro and Wei: Here we prove $\alpha(G) \geq \frac{\#V}{d(G)+1}$ for all graphs G, deducing Turán's theorem (theorem 2.14).
- Ajtai, Komlós, Szemerédi and Shearer: Here we prove $\alpha(G) \geq \frac{\log(\Delta(G))\#V}{\Delta(G)}$ for triangle-free graphs G.
- Campos, Jenssen, Michelen and Sahasrabudhe: Here we prove $\alpha(G) \geq \frac{\log(\Delta(G))\#V}{\Delta(G)}$ for graphs G with controlled codegrees.

3.1 The Caro-Wei Theorem

The best we can do for a graph with no structure is a classical result proved independently by Caro and Wei. Note in the greedy algorithm given in the proof of theorem 2.14 that we may delete far fewer than $\Delta(G) + 1$ vertices in some steps when $d(G) \ll \Delta(G)$. We ought to be able to replace $\Delta(G) + 1$ for $d(G) + 1$ by choosing a "good" ordering of V. Indeed, due to the independent work of Caro and Wei we can[23] [10] [4]. Our proof is a clean application of the probabilistic method, courtesy of Alon and Spencer [2].

Theorem 3.1. Let $G = (V, E)$ be a graph with n vertices. Then

$$\alpha(G) \geq \frac{n}{d(G) + 1}$$

Idea: Randomly permute the vertices of G and greedily choose vertices that have the property that none of their neighbours appear before them in this permutation.

Proof. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a random permutation of $V = (v_1, \ldots, v_n)$. We construct an independent set I by adding π_i to I has the property that no $\pi_j \in \mathcal{N}_G(\pi_i)$ for $j < i$. It is clear such a set will be independent. Furthermore, by basic combinatorics[24]

$$\mathbb{P}(v \in I) = \frac{1}{d_G(v) + 1}$$

Hence we have

$$\mathbb{E}[\#I] = \sum_{v \in V} \frac{n}{d_G(v) + 1} \geq \frac{n}{d(G) + 1}$$

via the lineaity of expectations and Jensen's inequality. Conclude with lemma 2.16. $\square$

[23] Finding a reference for Wei's proof is quite the challenge. According to Griggs [10] it was shown in his PhD thesis at the University of Hawaii, but I cannot find a copy of this anywhere.

[24] This is a classic problem concerning orderings of permutations $\sigma \in S_n$. One proof methodology is to induct. An alternative can be given by a direct count.

Remark 3.2. Caro and Wei both originally proved this result with clever inductions. Our proof is both far quicker and more instructive.

Remark 3.3. As $d(G) \leq \Delta(G)$ we have theorem 2.14 as an immediate corollary.

3.2 Shearer's Theorem

Suppose we now impose some structure onto our graph. Can we obtain a stronger bound on $\alpha(G)$ in $\#V(G)$ and $\Delta(G)$? Ajtai, Komlos and Szemeredi were first to obtain something in this direction, showing for triangle-free G we have $\alpha(G) = \Omega(n \log(\Delta(G))/\Delta(G))$ [1]. They were able to in particular show

$$\alpha(G) \geq 0.01 \frac{\log(\Delta(G)) \cdot \#V(G)}{\Delta(G)}$$

Their method is to construct[25] a large independent set inductively. We find groups $P \subset V(G)$ that have neighbourhoods of high degree compared to the degree of P.

Shearer was able to sharpen this with the following theorem[26] [30].

Theorem 3.4. Let $G = (V, E)$ be a triangle-free graph. Then

$$\alpha(G) \geq \frac{\Delta(G)\#V}{\Delta(G)}$$

We will use this theorem in section 4.2.

Shearer's approach was to consider a well chosen differential equation with solution f and to bound $\alpha(G) \leq \sum_{i=1}^{\#V} f(d_i)$ by induction. This argument is rather abstract and provides little intuition on the independence set problem on the whole. We elect to prove a version weaker by a constant factor, making no effort to optimise this constant. The following argument is due to Noga Alon[27]. Before giving this argument, we recall an elementary result regarding Binomial sums[28].

Lemma 3.5. Let $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

Idea: Use the Binomial theorem and calculus on $(1 + x)^n$.

[25] Their proof is only existence proof, but it is rather constructive in nature.

[26] This is a corollary of his theorem, which is stated rather esoterically.

[27] Originally given as a remark.

[28] A motivated high-school student could prove this, but it was decided that a statement and proof was warranted for readers of a non-combinatorial background.

Proof. By the Binomial Theorem, we have for $x \in \mathbb{R}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{16}$$

Differentiate equation (16) with respect to x to obtain

$$n(1+x)^n = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}$$

Set $x = 1$ to deduce the result. $\square$

Theorem 3.6. Let $G = (V, E)$ be a triangle free graph with maximimum degree Δ. Then

$$\alpha(G) \geq \frac{\log(\Delta)\#V}{32\Delta}$$

Proof. If $\Delta < 16$, then

$$\frac{\log(\Delta)\#V}{32\Delta} < \frac{\log(3)\#V}{512} < \frac{\#V}{16} \leq \frac{\#V}{\Delta+1}$$

so (3.6) holds. Instead suppose $\Delta \geq 16$. Sample W uniformly from the set of independent sets in G. For $v \in V$, define

$$X_v := \Delta \mathbf{1}\{v \in W\} + \#N_G(v) \cap W$$

We begin with the following claim

Claim. We have $\mathbb{E}[X_v] \geq \frac{1}{16}\log(\Delta)$

Proof of claim. Let H be the induced subgraph of G with vertices $V \setminus (\mathcal{N}_G(v) \cup \{v\})$. Fix an independent set U in H. It is clear there must be some choice of U such that $W \cap V(H) = U$. Hence, it suffices to show that

$$\mathbb{E}[X_v \mid W \cap \mathcal{N}_G(v) = U] \geq \frac{1}{16}\log(\Delta)$$

Let $T := \{v_0 \in \mathcal{N}_G(v) : v_0 \notin \cup_{u \in U} \mathcal{N}_G(u)\}$. If $v \in W$, there is only one choice for W, $W = U \cup \{v\}$. Otherwise, due to the triangle-free criteria, we can have any W of the form $W = U \cap T'$ where $T' \subset T$. As there are $2^{\#T}$ choices for W in the latter case, one in the former. Hence, writing $X_v = X_v \mathbf{1}\{v \in W\} + X_v \mathbf{1}\{v \notin W\}$, it is clear that

$$\mathbb{E}[X_v \mid W \cap \mathcal{N}_G(v) = U] = \frac{1}{2^{\#T}+1}\Delta + \frac{1}{2^{\#T}+1}\sum_{i=0}^{\#T} \binom{\#T}{i} i$$

Using lemma 3.5 this reduces to

$$\mathbb{E}[X_v \mid W \cap \mathcal{N}_G(v) = U] = \frac{\Delta}{2^{\#T}+1} + \frac{2^{\#T-1}\#T}{2^{\#T}+1} \geq \frac{1}{16}\log(\Delta)$$

where the inequality follows from casework on $2^{\#T+1} \leq \sqrt{\Delta}$. □

Define $X := \sum_{v\in V} X_v$. Our claim and the linearity of expectations together show

$$\mathbb{E}[X] \geq \frac{\log(\Delta)\#V}{16}$$

Also, it is clear we may bound

$$\mathbb{E}[X] \leq 2\Delta\mathbb{E}[\#W]$$

as each $w \in W \subseteq V$ contributes at most $\Delta + d_G(w) \leq 2\Delta$ to the sum. Deduce

$$\mathbb{E}[\#W] \geq \frac{\log(\Delta)\#V}{32\Delta}$$

Apply lemma 2.16 to conclude the proof. □

3.3 The Campos-Jennsen-Michelen-Sahasrabudhe Theorem

We will spend this section proving the following theorem of Campos, Jenssen, Michelen and Sahasrabudhe. We will use this to exhibit the existence of packings with density $(1-o(1))2^{-d-1}d\log d$ in section 4.

Theorem 3.7. Let $G = (V,E)$ be a graph with $\#V(G) = n$ and let $\Delta \geq 160^{160}$ be such that $\Delta(G) \leq \Delta$ and $\Delta_{\text{co}} \leq 2^{-7}\Delta(\log\Delta)^{-7}$. Then

$$\alpha(G) \geq \left(1 - \frac{40\log\log\Delta}{\log\Delta}\right)\frac{n\log\Delta}{\Delta}$$

In particular,

$$\alpha(G) \geq (1-o(1))\frac{n\log\Delta}{\Delta}$$

where $o(1) \to 0$ as $\Delta(G) \to \infty$.

Let $I = \emptyset$. Our strategy is as follows:

1. Regularise: Greedily add edges to G until it is "almost regular".
2. Nibble: Remove a random subset $A \subseteq V(G)$ and its neighbours from G.
3. Extract: Find a maximally independent set I_A in A and append $I = I \cup I_A$.

4. Recurse: Repeat steps 1 through 3 until I is sufficiently large.

Let $G, G' = (V, E), (V', E')$ with $V = V'$ and $E' \supseteq E$. It is clear that if I is an independent set in G' then it must also be independent in G. It is also clear that if we have an independent set $I \subseteq A$ and an independent set $J \subseteq V \setminus (A \cup \mathcal{N}_G(A))$ then $I \cup J$ is an independent set in G. Hence our set I described above will be independent in G.

Let G be a graph and $A \subseteq V$. We denote the induced subgraph with vertex set A by $G[A]$. We say G is Δ-*focused* if $d_G(v) \in \{\Delta - 1, \Delta\}$ for all $v \in V$. A *p-random subset* of a set X is a random subset $Y \subseteq X$ where each $x \in X$ is in Y with probability p independently of whether any of the other $X \ni x' \neq x$ were kept in Y. For $u, v \in V(G)$ define

$$\operatorname{dist}_G(u, v) := \min\{n \in \mathbb{N}_0 : \exists \text{ path } u \to v \text{ length } n\}$$

and for $v \in V(G)$ define

$$\mathcal{N}_G^{(k)}(v) := \{u \in V(G) : \operatorname{dist}_G(u, v) \leq k\}$$

We will need the following technical lemma from [3]. We disclude its proof for the sake of brevity, see section 3 in [3] for the details.

Lemma 3.8. Let $\Delta \geq 1$. Let $\gamma \in [0, 1/2]$, $\eta \in [\Delta^{-1/2}, \gamma^2/8]$ and $\alpha \in [2\gamma^2, \gamma]$. Let G be Δ-focused with $\Delta_{\mathrm{co}}(G) \leq \eta\Delta$. Let A be a $\frac{\gamma}{\Delta}$-random subset of $V(G)$ and set $G' = G[V(G) \setminus (A \cup \mathcal{N}_G(A))]$. Then, for all $v \in V(G)$,

$$\mathbb{P}(d_{G'}(v) \geq (1 - \gamma + \alpha) d_G(v) | v \in V(G')) \leq \exp\left(-\frac{\alpha^2}{32\gamma\eta}\right)$$

and for all distinct $u, v \in V(G)$

$$\mathbb{P}(d_{G'}(u, v) \geq (1 - \gamma + \alpha)\eta\Delta | u, v \in G') \leq \exp\left(-\frac{\alpha^2}{32\gamma\eta}\right)$$

These upper tail inequalities will be used to show that there is a large subgraph with appropriately shrunk degrees and codegrees after our "nibble" step.

Nibbling on Δ-Focused Graphs

Our task now is to prove lemma 3.9 by applying lemma 3.8. This lemma will be used to show step 2 of our algorithm leaves us with a sufficiently nice subgraph and sufficiently many vertices to add to our independent set.

Lemma 3.9. Let $\Delta \geq 2^{11}$, $8\Delta^{-\frac{1}{8}} \leq \gamma \leq \frac{1}{2}$ and $n \geq \Delta^4$. Let G be a Δ-focused graph with n vertices and $\Delta_{\text{co}}(G) \leq 2^{-6}\gamma^3\Delta(\log\Delta)^{-1}$. Let $\alpha \geq 2\gamma^2$. Then $\exists A \subset V(G)$ and $\exists C \subset V(G) \setminus (A \cup \mathcal{N}_G(A))$ with A satisfying

$$\#A \geq \frac{\gamma}{\Delta}(1-\alpha)n \quad \text{and} \quad \#E(G[A]) \leq \frac{\gamma}{\Delta} \cdot \gamma n \tag{17}$$

and C satisfying

$$\#C \geq (1-\gamma-\alpha)n, \ \ \Delta(G[C]) \leq (1-\gamma+\alpha)\Delta \ \text{ and } \ \Delta_{\text{co}}(G[C]) \leq (1-\gamma+\alpha)\eta\Delta \tag{18}$$

where $\eta := \frac{1}{\Delta}\max\{\Delta_{\text{co}}(G), 2\sqrt{\Delta}\}$.

Idea: Take A to be a $\frac{\gamma}{\Delta}$-random subset of $V(G)$ and obtain C from $V(G) \setminus (A \cup \mathcal{N}_G(A))$ by deleting vertices with too high degree/codegree.

Proof. Let $A \subseteq V(G)$ be a $\frac{\gamma}{\Delta}$-random subset. By lemma 2.16, we have[29]

$$\mathbb{P}(\#A \geq \mathbb{E}[\#A]\ ;\ \#E(G[A]) \leq \mathbb{E}[E(G[A])]\ ;\ \#A \cup \mathcal{N}_G(A) \leq \mathbb{E}[\#A \cup \mathcal{N}_G(A)]) > 0$$

We compute $\mathbb{E}[\#A] = n \cdot \frac{\gamma}{\Delta} \geq (1-\alpha)\frac{n\gamma}{\Delta}$ with the linearity of expectations. Compute $\mathbb{E}[\#E(G[A])] = \#E(G) \cdot \left(\frac{\gamma}{\Delta}\right)^2 \leq n\frac{\gamma^2}{\Delta}$ analogously. As G is Δ-focused, we may bound $\mathbb{E}[\#A \cup \mathcal{N}_G(A)] \leq \left(\frac{\gamma}{\Delta}\right) n\Delta \leq (\gamma + \alpha/2)n$. Hence $\exists A \subseteq V(G)$ with

$$\#A \geq (1-\alpha)\frac{n\gamma}{\Delta},\ \#E(G[A]) \leq \frac{\gamma^2 n}{\Delta} \ \text{ and } \ \#A \cup \mathcal{N}_G(A) \leq \left(\gamma + \frac{\alpha}{2}\right)n$$

Let $G' = G[V(G) \setminus (A \cup \mathcal{N}_G(A))]$ and define $B \subseteq V(G')$ by

$$B := \{u : d_{G'}(u) \geq (1-\gamma+\alpha)\Delta\} \cup \{u : \exists v \in V(G') \text{ s.t. } d_{G'}(u,v) \geq (1-\gamma+\alpha)\eta\Delta\} \tag{19}$$

Set $C := V(G') \setminus B$. It is clear $\Delta(G[C]) \leq (1-\gamma+\alpha)\Delta$ and $\Delta_{\text{co}}(G[C]) \leq (1-\gamma+\alpha)\eta\Delta$. Hence, we're done if we prove $\#C \geq (1-\gamma-\alpha)n$. Observe

$$\#C = \#V(G) - \#A \cup \mathcal{N}_G(A) - \#B \geq \left(1 - \gamma - \frac{\alpha}{2}\right)n - \#B$$

Thus, it suffices to prove $\#B \leq \frac{\alpha}{2}n$. Write $B = B_1 \cup B_2$ where B_1 (resp. B_2) is the left (resp. right) side of the intersection in equation (19). We will bound $\#B_1, \#B_2 \leq \frac{\alpha n}{4}$ using lemma 3.8. Observe[30] $\eta \in [\Delta^{-1/2}, \gamma^2/8]$ so that all our parameters satisfy the conditions given in lemma 3.8. Hence, by the linearity of expectations,

$$\mathbb{E}[\#B_1] \leq \sum_{u \in V(G)} \mathbb{P}(d_{G'}(u) \geq (1-\gamma+\alpha)\Delta | u \in V(G')) \leq n\exp\left(-\frac{\alpha^2}{32\gamma\eta}\right)$$

[29] We use a slight generalisation here to vector valued random-variables.

[30] This isn't particularly hard to see, the lower bound comes from the $\sqrt{\Delta}$ in the minimum term for η, the ubber bound from the $\Delta_{\text{co}}(G)$ term.

Suppose $\eta = \Delta_{\text{co}}(G)/\Delta$. Then

$$\Delta_{\text{co}}(G) \le 2^{-6}\gamma^3\Delta(\log\Delta)^{-1} \text{ and } \alpha \ge 2\gamma^2 \Longrightarrow \frac{\alpha^2}{32\gamma\eta} \ge 8\log\Delta$$

Now suppose $\eta = 2\Delta^{-1/2}$. Then

$$\gamma \ge 8\Delta^{-1/8} \text{ and } \alpha \ge 2\gamma^2 \Longrightarrow \frac{\alpha^2}{32\gamma\eta} \ge 2^6\Delta^{-1/8}$$

Then as $\Delta \ge 2^{11}$, it is clear that in either case $\exp\left(-\frac{\alpha^2}{32\gamma\eta}\right) \le \frac{\alpha}{4}$ so that $\mathbb{E}[\#B_1] \le \frac{\alpha n}{16}$. Now we work with $\mathbb{E}[\#B_2]$. Suppose $\Delta_{\text{co}}(G) \le \sqrt{\Delta}$. Then $\eta\Delta = 2\sqrt{\Delta} > \Delta_{\text{co}}(G)$ and hence $\mathbb{E}[\#B_2] = 0 \le \frac{\alpha n}{16}$. Now suppose $\Delta_{\text{co}}(G) > \sqrt{\Delta}$. Observe,

$$\begin{aligned}
&\mathbb{P}(\exists v \in V(G') \text{ s.t. } d_{G'}(u,v) \ge (1-\gamma+\alpha)\eta\Delta) \\
&\le \sum_{v\in\mathcal{N}_G(\mathcal{N}_G)} \mathbb{P}(d_{G'}(u,v) \ge (1-\gamma+\alpha)\eta\Delta | u,v \in V(G')) \\
&\le \Delta^2 \max_{u,v\in V(G)} \mathbb{P}(d_{G'}(u,v) \ge (1-\gamma+\alpha)\eta\Delta | u,v \in V(G')) \\
&\le \Delta^2 \exp\left(-\frac{\alpha^2}{32\gamma\eta}\right) \le \frac{\alpha}{4}
\end{aligned}$$

Hence, summing over $v \in V(G)$, we deduce $\mathbb{E}[\#B_2] \le \frac{n\alpha}{4}$. Applying lemma 2.16, we deduce $\#B \le \frac{\alpha n}{2}$ and the lemma is established. ☐

Regularisation of Graphs

Now we must show that we can add edges to a graph to make it "almost-regular" without increasing the codegree.

Lemma 3.10. Let $\Delta \ge 2$. Let G be a graph with $n \ge 2\Delta^4$ vertices and $\Delta(G) \le \Delta$. Then $\exists\overline{G}$ with $\overline{G}$ $(\Delta+1)$-focused, $V(\overline{G}) = V(G)$, $E(\overline{G}) \supseteq E(G)$ and $\Delta_{\text{co}}(\overline{G}) \le \max\{1, \Delta_{\text{co}}(G)\}$.

Idea: Greedily add edges between vertices of low degree that are far apart.

Proof. Let $G_0 := G$ and let $T < T'$ be constants to be determined later. Construct $G_{i+1} = G_i + e_{i+1}$ where $e_{i+1} \in E(K_{\#V(G)})$ is an edge we'll specify later. Define

$$\mathcal{D}_i := \{v \in V(G_i) : d_{G_i}(v) < \Delta\} \quad \text{and} \quad \mathcal{S}_i := \{v \in V(G_i) : d_{G_i}(v) \le \Delta\}$$

If there exist distinct $u, v \in \mathcal{D}_i$ with $\text{dist}_{G_i}(u,v) \ge 4$ set $e_{i+1} = \{u,v\}$. If there do not exist such u, v conclude at step $i = T$. For $i > T$, if there exist distinct $u \in \mathcal{D}_i$, $v \in \mathcal{S}_i$ with $\text{dist}_{G_i}(u,v) \ge 4$ then set $e_{i+1} = \{u,v\}$. Otherwise conclude at step $i = T'$ and set $\overline{G} = G_{T'}$. It is clear that $\overline{G}$ is $(\Delta+1)$-focused if and only if $D_{T'} = \emptyset$.

Claim. Our algorithm terminates with $\mathcal{D}_{T'} = \emptyset$.

Proof of claim. At step T we have $\mathrm{dist}_{G_T}(u,v) \leq 3$ for all $u, v \in \mathcal{D}_T$ by definition, so it is clear $\#\mathcal{D}_T \leq \Delta^3$. Further, it is also clear that for $i \geq T$ we have $\#S_{i+1} \geq \#S_i - 1$ and

$$\sum_{v \in \mathcal{D}_{i+1}} (\Delta - d_{G_{i+1}}(v)) \leq \left(\sum_{v \in \mathcal{D}_i} (\Delta - d_{G_i}(v)) \right) - 1 \tag{20}$$

Using $\mathcal{D}_T \leq \Delta^3$, it is clear

$$\sum_{v \in \mathcal{D}_T} (\Delta - d_{G_T(v)}) \leq \Delta^4 \tag{21}$$

Combining equations (20) and (21), we deduce that

$$\sum_{v \in \mathcal{D}_{T+j}} (\Delta - d_{G_{T+j}}) \leq \Delta^4 - j$$

for $j \geq 0$. Evaluating at $j = \Delta^4$ we see $D_{T+\Delta^4} = \emptyset$. Hence, $T' - T \leq \Delta^4$ as our algorithm terminates if $D_i = \emptyset$. For $i \leq T$, we only increase the degree of vertices with degree strictly less than Δ, so $\#S_T = \#V(G) \geq 2\Delta^4$ and for each $i \in \{T, \ldots, T + \Delta^4\}$ we have $\#S_i \geq \Delta^4$. Hence, for i as before and $v \in \mathcal{D}_i$, $\#S_i \setminus \mathcal{N}^{(3)}(v) \geq \#S_i - \Delta^3 > 0$. It is thus clear our algorithm terminates with $\mathcal{D}_{T'} = \emptyset$. □

All that remains is to check $\Delta_{\mathrm{co}}(\overline{G}) \leq \max\{1, \Delta_{\mathrm{co}}(G)\}$. By way of contradiction suppose there exist distinct u, v and $i \in \{0, \ldots, T' - 1\}$ with $d_{G_{i+1}}(u,v) \geq d_{G_i}(u,v) + 1$ and $d_{G_i}(u,v) > 0$. Then our chosen edge $e_{i+1} = \{u', v'\}$ must be so that $\mathrm{dist}_{G_i}(u', v') \leq 3$, which is obviously contradictory. □

Proof of Theorem 3.7

Let us begin with a sketch, adding some additional detail to our original sketch.

1. Choose large $\Delta \geq \Delta(G)$ and parameters meeting the conditions of lemma 3.9.

2. Construct $G = G_0 \to \overline{G_0} \to G_1 \to \overline{G_1} \to \cdots \to G_T$ for some large T by applying lemma 3.10 for $G_i \to \overline{G_i}$ and setting $G_{i+1} = \overline{G_i}[C]$ where C is given by lemma 3.9.

3. Show that the degree and codegrees of the G_i decrease appropriately for each i by induction and that $V(G_i)$ is sufficiently large.

4. Extract independent $I_i \subset A_i$ where A_i is given by the i^{th} application of lemma 3.9 by considering the isolated vertices in $G[A_i]$.

5. Take $I = \cup_{i=1}^{T} I_i$ and prove $\#I \geq (1 - o(1)) \frac{\log(\Delta(G))n}{\Delta(G)}$.

Proof of theorem 3.7. Fix $\epsilon \in (0, 1/2)$ and choose $\Delta_\epsilon \geq \exp\left(\frac{80}{\epsilon}\log\frac{80}{\epsilon}\right) \geq \Delta(G)$. Let $\gamma = (\log \Delta_\epsilon)^{-2}$, $\alpha = 2\gamma^2$ and $q = 1 - \gamma + 2\alpha$. Initialise $G_0 := G$ and let

$$T := \gamma^{-1}\left(\log \Delta_\epsilon - 32 \log\log \Delta_\epsilon - 64\right), \ \Delta_i := \lceil q^i(\Delta(G)+1)\rceil \ \text{ and } \ \Delta_i' := q^i \Delta_{\text{co}}(G)$$

Given $G_i, i \geq 0$ we obtain G_{i+1} by first applying lemma 3.10 to obtain a Δ_i-regular graph $\overline{G_i}$ and then applying lemma 3.9 to get $A_{i+1} \subseteq V(\overline{G_i})$ and $C_{i+1} \subseteq V(\overline{G_i}) \setminus (A \cup \mathcal{N}_{\overline{G_i}}(A))$, from which we define $G_{i+1} := \overline{G_i}[C_{i+1}]$. We repeat this process T times to obtain a sequence of disjoint vertex subsets $A_1, \ldots, A_T$. We will conclude by taking the union of large independent sets in each A_i, found by considering the isolated vertices.

Note that by adding disjoint copies of G to G, the ratio $\alpha(G)/\#V(G)$ is preserved. Hence WLOG we may assume[31]

$$(q - 3\alpha)^T n \geq 2(\Delta_\epsilon + 1)^4 \tag{22}$$

We begin by showing our process decreases the degree/codegree aptly while maintaining a reasonable size of the $\#V(G_i)$. First, we observe that

$$\Delta_i \geq \Delta_T \geq 2^{11} \quad \text{and} \quad \Delta_i^{1/8} \leq \gamma \leq \frac{1}{2} \tag{23}$$

for each $i \leq T$. By (22) and (23) we see that, when applying lemma 3.9, we need only check that $\Delta_{\text{co}}(\overline{G_i})$ is sufficiently small. To that end, observe that[32]

$$\Delta_i \geq \Delta_T \geq (4 \log \Delta(G))^{16} \tag{24}$$

for each $i \leq T$.

Claim 1. For each $0 \leq i \leq T$ we have $\Delta(G_i) \leq \Delta_i - 1$, $\overline{G_i}$ Δ_i-focused, both

$$\Delta_{\text{co}}(G_i) \leq \max\{\Delta_i', 2\sqrt{\Delta_i}\} \quad \text{and} \quad \Delta_{\text{co}}(\overline{G_i}) \leq \max\{\Delta_i', 2\sqrt{\Delta_i}\}$$

and $\#V(G_i) \geq (q - 3\alpha)^i n$.

Proof of claim. We work by induction, noting the claims trivially hold for G_0. By (22) we have $n \geq 2(\Delta(G)-1)^4$ and hence we can apply lemma 3.10 with parameter $\Delta_0 = \Delta(G)-1$ to obtain $\overline{G_0}$ meeting the necessary conditions. This completes the base case.

Now suppose the claims hold for G_i, $i < T$. By the inductive hypothesis $\#V(G_i) \geq (q - 3\alpha)^i n$ and so by equation (22) we have $V(G_i) \geq 2(\Delta(G)+1)^4 \geq 2(\Delta_i - 1)^4$. Hence, may apply lemma 3.10 with parameter $\Delta_i - 1$ to get Δ_i-focused $\overline{G_i}$ with $V(\overline{G_i}) = V(G_i)$,

[31] Our final bound will have a factor of n, so we may replace our large n with the true $n = \#V(G)$ safely.
[32] Both (23) and (24) follow from some crude bounding. We omit the details as they're uninteresting and are (somewhat) easily verified.

$E(\overline{G_i}) \supseteq E(G_i)$ and $\Delta_{\text{co}}(\overline{G_i}) \le \max\{\Delta_{\text{co}}(G_i), 1\} \le \max\{\Delta_i', 2\sqrt{\Delta_i}\}$ where the final inequality is immediate from the inductive hypothesis.

To apply lemma 3.9 to $\overline{G_i}$ with parameters Δ_i, γ, α we require $\Delta_{\text{co}}(\overline{G_i}) \le 2^{-6}\gamma^3\Delta_i(\log\Delta_i)^{-1}$. Suppose $\Delta_i' \ge 2\sqrt{\Delta_i}$. Then

$$\Delta_{\text{co}}(\overline{G_i}) \le q^i\Delta_{\text{co}}(G) \le (q^i\Delta(G)) \cdot (\Delta_{\text{co}}(G)/\Delta(G)) \le 2^{-6}\Delta_i\gamma^3(\log\Delta_i)^{-1}$$

where the final inequality follows from $\Delta_i \ge q^i\Delta(G)$ and $\Delta_{\text{co}}(G) \le 2^{-7}\Delta(G)(\log\Delta(G))^{-7}$. Conversely, suppose $\Delta_i' \le 2\sqrt{\Delta_i}$. Then, by (24),

$$\Delta_{\text{co}}(\overline{G_i}) \le 2\sqrt{\Delta_i} = 2(\Delta_i)(\Delta_i)^{-1/2} \le 2\Delta_i(2\log\Delta(G))^{-7} \le 2^{-6}\Delta_i\gamma^3(\log\Delta_i)^{-1}$$

and in either case we have $\Delta_i' \le 2^{-6}\Delta_i\gamma^3(\log\Delta_i)^{-1}$. Hence, it is clear we can apply lemma 3.9 to get $A_i \subseteq V(\overline{G_i}), C \subseteq V(\overline{G_i}) \setminus (A_i \cup \mathcal{N}_{\overline{G_i}}(A_i))$ obeying (17) and (18) respectively. Set $G_{i+1} = G_i[C_i]$ to deduce that the claim holds true for $i+1$. $\square$

Now we find a large independent set in each of the A_i “nibbled” off of G_i.

Claim 2. For each $1 \le i \le T$, there is $I_i \subset A_i$ that is independent in G with

$$\#I_i \ge (1 - 8\gamma^{1/2})\gamma n/\Delta_\epsilon \tag{25}$$

Proof of claim. During our nibble steps in the proof of claim 1, we obtained a sequence of disjoint $A_i \subset V(G), 0 \le i \le T-1$ with

$$\#A_i \ge (1-\alpha)\gamma\#V(G_i)/\Delta_i \quad \text{and} \quad \#E(G[A_i]) \le \gamma^2\#V(G_i)/\Delta_i$$

Deduce

$$\#E(G[A_i]) \le (1-\alpha)^{-1}\gamma\#A_i \le 2\gamma A_i$$

Hence[33] we must have at least $1-4\gamma$ isolated vertices. Let I_i be the collection of these vertices. Note clearly each I_i is independent in G. Then

$$\#I_i \ge (1-4\gamma)\#A_i \ge (1-4\gamma)(1-\alpha)\gamma\#V(G_i)/\Delta_i$$

Using claim 1, deduce

$$\#I_i \ge (1-4\gamma)(1-\alpha)\gamma(q-3\alpha)^i\gamma n/\Delta(G) \ge (1-4\gamma)(1-3\alpha)^{i+1}\gamma n/\Delta_\epsilon$$

Bounding $(1-3\alpha)^{i+1} \ge (1-7\gamma^{1/2})$ completes the proof. $\square$

[33] By e.g. applying the handshaking lemma

Let $I = \cup_{i=0}^{T} I_i$. We deduce with claim 2 that

$$\#I \geq T(1 - 8\gamma^{1/2})\gamma n/\Delta_\epsilon$$

Now we bound[34]

$$T(1 - 8\gamma^{1/2})\gamma \geq (1 - \epsilon) \log \Delta_\epsilon$$

With the following claim[35] we can immediately deduce theorem 3.7.

Claim 3. We have

$$\epsilon \geq \frac{40 \log \log \Delta_\epsilon}{\log \Delta_\epsilon}$$

and in particular $\epsilon = o(1)$ where $o(1) \to 0$ as $\Delta(G) \to \infty$.

Proof of claim. Set $x = \frac{80}{\epsilon} \geq 160$ and let W denote the Lambert W function[36]. Then

$$\log(\Delta_\epsilon) \geq x \log x \implies x \leq \frac{W(\log \Delta_\epsilon)}{\log \Delta_\epsilon}$$

In [14], it shown $W(t) \geq \log t - \log \log t$. Hence,

$$\epsilon \geq \frac{80}{\log \Delta_\epsilon} [\log \log \Delta_\epsilon - \log \log \log \Delta_\epsilon]$$

Noting $\Delta_\epsilon \geq e^{e^2}$ for each $\epsilon \in (0, 1/2)$, we deduce

$$\epsilon \geq \frac{40 \log \log \Delta_\epsilon}{\log \Delta_\epsilon} \to 0 \text{ as } \Delta_\epsilon \to \infty$$

establishing the claim. □

Using claim 3, theorem 3.7 is established. □

Remark 3.11. The requirement $\Delta \geq 160^{160}$ is not given in [3], but it is needed for our proof. It appears to be implicitly used in the proof of their $\theta(d)$ bound, but it is once again not made explicit.

[34] Again, this follows by crude, uninteresting, bounding.
[35] This claim is stated without proof in [3]. I imagine our proof is what they had in mind.
[36] This is the inverse of $x \mapsto xe^x, x > \frac{1}{e}$. This function is classical, dating back to Euler. It is easy to see this function is monotone increasing.

4 Lower Bounds For $\theta(d)$

Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body. For $L > 0$ define the *L-dilated sphere packing density* of $\mathcal{C}$ by

$$\theta_{\mathcal{C}}(d; L) := \sup_{\mathcal{P}} \mathcal{D}(\mathcal{P} | L\mathcal{C})$$

where $L\mathcal{C}$ is the L-dilation of $\mathcal{C}$, $\mathcal{D}(\mathcal{P}|L\mathcal{C})$ denotes the relative density of $\mathcal{P}$ in $L\mathcal{C}$ and the supremum is taken over sphere packings of $L\mathcal{C}$. By lemma 2.8,

$$\theta(d) = \limsup_{L \to \infty} \theta_{\mathcal{C}}(d; L)$$

for any convex body $\mathcal{C} \subset \mathbb{R}^d$.

In this section we prove $\theta(d) = \Omega(2^{-d})$, $\theta(d) = \Omega(2^{-d} d)$ and $\theta(d) = \Omega(2^{-d} d \log d)$. All proofs follow the same general structure, which was motivated in section 1.3.

1. Sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$ for some well chosen intensity λ.
2. Prove that in expectation $\mathbf{X}$ does not contain too many "bad" points $B \subset \mathbf{X}$.
3. Show the geometric graph $G(\mathbf{X} \setminus B)$ has a large independent set.

The bulk of each proof comes from step 2. We decide what our "bad" points are based on the conditions that need to met by the left over geometric graph in order to apply our independence number bound in step 3. To prove there aren't too many points, we compute the expected number of bad points with the Mecke formula and apply the first moment method to get a collection with no bad points.

4.1 $\theta(d) = \Omega(2^{-d})$

Using the concentration of Poisson random variables about their mean, we give the following sketch argument. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and let $L > 0$. Let $\lambda > 0$ and sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$. Let $G := G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$.

(i) Observe that $\#\mathbf{X} \approx \lambda \mathrm{Vol}(L\mathcal{C})$ and that for each $x \in \mathbf{X}$,

$$d_G(x) = \#B(x; 2r_d) \cap (\mathbf{X} \setminus \{x\}) \approx \lambda \mathrm{Vol}(B_x(2r_d)) = 2^d \lambda$$

(ii) Hence $\Delta(G) \approx 2^d \lambda$ and by Turán's theorem

$$\alpha(G) \gtrapprox \frac{\lambda \mathrm{Vol}(L\mathcal{C})}{2^d \lambda + 1}$$

(iii) Deduce

$$\theta_{\mathcal{C}}(d; L) \lessapprox \frac{\lambda}{2^d\lambda + 1} \approx 2^{-d}$$

and taking $L \to \infty$ gives $\theta(d) \lessapprox 2^{-d}$.

This sketch leads us to believe packings of density $\Omega(2^{-d})$ are far from rare. We may formalise this by looking at either the *entropy density* and *pressure* in the *canonical hard-sphere model*[37]. In [22] a non-trivial bound for the entropy density of sphere packings $\Omega(d2^d)$ and the pressure in the canonical hard sphere model. As a corollary, they are able to prove $\theta(d) \geq (1 + o(1)) \log\left(\frac{2}{\sqrt{3}}\right) d2^{-d}$, where $\log\left(\frac{2}{\sqrt{3}}\right) \approx 0.144$ beats our constant in theorem 4.8 by a factor of approximately five.

Our first lemma gives a bound on the number of points with "high degree".

Lemma 4.1. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and $L > 0$. Let $\lambda > 0$ and sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$. Let $G := G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$. Let $\Delta > 0$. Then for all $x \in \mathbf{X}$ we have

$$\mathbb{E}[\#\{x \in \mathbf{X} : d_G(x) > \Delta\}] \leq \lambda \mathrm{Vol}(L\mathcal{C})\epsilon_1(d, \lambda, \Delta)$$

where

$$\epsilon_1(d, \lambda, \Delta) := \begin{cases} \exp\left(-\frac{1}{2^d\lambda}(\Delta - 1 - 2^d\lambda)^2\right) & : \Delta \in (2^d\lambda + 1, 2^{d+1}\lambda + 1) \\ \exp\left(2^d\lambda - \Delta + 1\right) & : \Delta \notin (2^d\lambda + 1, 2^{d+1}\lambda + 1) \end{cases} \tag{26}$$

Proof. It is clear that

$$\mathbb{E}\left[\#\{x \in \mathbf{X} : d_G(x) > \Delta\}\right] = \mathbb{E}\left[\#\{x \in \mathbf{X} : \#B(x; 2r_d) \cap \mathbf{X} \geq \Delta\}\right]$$

By Mecke's equation (theorem 2.13), the right hand side is equal to

$$\lambda \int_{L\mathcal{C}} \mathbb{P}(\#B(x; 2r_d) \cap (\mathbf{X} \cup \{x\}) \geq \Delta)\mathrm{d}x$$

As $\mathbf{X}$ has measure zero[38] this is integral is equal to

$$\lambda \int_{L\mathcal{C}} \mathbb{P}(\#B(x; 2r_d) \cap \mathbf{X} \geq \Delta - 1)\mathrm{d}x$$

The random variable $\#B(x; 2r_d) \cap \mathbf{X}$ has $\mathrm{Po}(\mathrm{Vol}(B(x; 2r_d) \cap L\mathcal{C})\lambda)$ distribution. As $B(x; 2r_d) \cap L\mathcal{C} \subseteq B(x; 2r_d)$ and $\mathrm{Vol}(B(x; 2r_d)) = 2^d$ it is clear that $\#B(x; 2r_d) \cap \mathbf{X}$ dominated by an independent $\mathrm{Po}(2^d\lambda) =: Z$ random variable. Hence, we may bound

$$\lambda \int_{L\mathcal{C}} \mathbb{P}(\#B(x; 2r_d) \cap \mathbf{X} \geq \Delta - 1)\mathrm{d}x \leq \lambda \mathrm{Vol}(L\mathcal{C})\mathbb{P}(Z \geq \Delta - 1)$$

[37] The definition of these quantities lies outside the scope of this thesis.

[38] Under the Lebesgue measure in $\mathbb{R}^d$ that is.

We now bound this $\mathbb{P}(Z \geq \Delta - 1)$ term. Observe,

$$\mathbb{P}(Z \geq \Delta - 1) = \mathbb{P}\left(Z - 2^d\lambda \geq 2^d\lambda\left(\frac{\Delta - 1}{2^d\lambda} - 1\right)\right)$$

and hence deduce, via lemma 2.10,

$$\mathbb{P}(Z \geq \Delta - 1) \leq \exp\left(-2^d\lambda \min\left\{\frac{\Delta - 1}{2^d\lambda} - 1, \left(\frac{\Delta - 1}{2^d\lambda} - 1\right)^2\right\}\right)$$

With the following simple calculations, see

$$\frac{\Delta - 1}{2^d\lambda} - 1 \leq \left(\frac{\Delta - 1}{2^d\lambda} - 1\right)^2 \iff (\Delta - 1)^2 - 3 \cdot 2^d\lambda(\Delta - 1) + 2 \cdot (2^d\lambda)^2 \geq 0$$
$$\iff (\Delta - 1 - 2^d\lambda)(\Delta - 1 - 2^{d+1}\lambda) \geq 0 \iff \Delta \notin (2^d\lambda + 1, 2^{d+1}\lambda + 1)$$

Hence,

$$\exp\left(-2^d\lambda \min\left\{\frac{\Delta - 1}{2^d\lambda} - 1, \left(\frac{\Delta - 1}{2^d\lambda} - 1\right)^2\right\}\right)$$
$$= \underbrace{\begin{cases} \exp\left(-\frac{1}{2^d\lambda}(\Delta - 1 - 2^d\lambda)^2\right) & : \Delta \in (2^d\lambda + 1, 2^{d+1}\lambda + 1) \\ \exp\left(2^d\lambda - \Delta + 1\right) & : \Delta \notin (2^d\lambda + 1, 2^{d+1}\lambda + 1) \end{cases}}_{=\epsilon_1(d,\lambda,\Delta)}$$

With this equality, the lemma follows immediately. □

Theorem 4.2. For $d > 1$,

$$\theta(d) \geq (1 - \exp(-2^d))2^{-d} = (1 - o(1))2^{-d}$$

where $o(1) \xrightarrow{d\to\infty} 0$.

Proof. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and $L > 0$. Fix $\delta \in (0, 1)$, set $\lambda = \delta^{-2}$ and sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$. Let $\Delta = (1 + \delta)2^d\lambda + 1$. Let $G := G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$. Let $B := \{x \in \mathbf{X} : d_G(x) > \Delta\}$ be the vertices of G with degree greater than Δ. With lemma 4.1, bound

$$\mathbb{E}[\mathbf{X} \setminus B] \geq \mathbb{E}[\mathbf{X}] - \mathbb{E}[B] \geq \lambda\mathrm{Vol}(L\mathcal{C})(1 - \epsilon_1(d, \lambda, \Delta))$$

where ϵ_1 is given in equation (26). Evaluating at our chosen λ and Δ, deduce

$$\mathbb{E}[\mathbf{X} \setminus B] \geq \delta^{-2}\mathrm{Vol}(L\mathcal{C})(1 - \exp(-2^d))$$

Thus, the induced geometric graph constructed by deleting vertices in B has maximal degree at most Δ and in expectation $\delta^{-2}\mathrm{Vol}(L\mathcal{C})(1-\exp(-2^d))$ vertices. Hence, by lemma 2.16, we have the existence of $\mathbf{Y} \subset \mathbb{R}^d$ with $H := G(\mathbf{Y}, 2r_d)$ satisfying $\Delta(H) \leq \Delta$ and $\#\mathbf{Y} \geq \delta^{-2}\mathrm{Vol}(L\mathcal{C})(1-\exp(-2^d))$. By theorem 2.14, this graph has independence number $\alpha(H)$ satisfying

$$\alpha(H) \geq \frac{\delta^{-2}\mathrm{Vol}(L\mathcal{C})(1-\exp(-2^d))}{(1+\delta)2^d\delta^{-2}+1}$$

Using equation (2), deduce

$$\theta_{\mathcal{C}}(d;L) \geq \frac{\delta^{-2}(1-\exp(-2^d))}{(1+\delta)2^d\delta^{-2}+1} = \frac{1-\exp(-2^d)}{(1+\delta)2^d+\delta^2}$$

Define the map

$$f_d : [0,1) \to \mathbb{R},\ \delta \mapsto \frac{1-\exp(-2^d)}{(1+\delta)2^d+\delta^2}$$

We have seen that $\theta_{\mathcal{C}}(d;L) \geq f_d(\delta)$ for all $\delta \in (0,1)$ and it is clear that f_d is continuous on $[0,1)$. Hence, by taking limits

$$\theta_{\mathcal{C}}(d;L) \geq \lim_{\delta\to 0} f_d(\delta) = f_d(0) = (1-\exp(-2^d))2^{-d}$$

Taking $L \to \infty$, deduce

$$\theta(d) \geq (1-\exp(-2^d))2^{-d}$$

which completes the proof. □

Remark 4.3. Our bound converges to the trivial bound at a super-exponential rate as $d \to \infty$. Asymptotically, this is the best we expect to be able to do, using homogenous Poisson proceeses, with theorem 2.14. While me may take our process to be arbitrarily dense, we must have $\Delta \geq (1+o(1))2^d\lambda$ so that $\mathcal{D}(\mathcal{P}, L\mathcal{C}) \geq (1-o(1))2^{-d}$.

4.2 $\theta(d) = \Omega(2^{-d}d)$

Let us again begin with a sketch argument using the concentration of Poisson random variables about their mean. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and let $L > 0$. Let $\lambda > 0$ and sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$. Let $G := G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$. Throughout the sketch c denotes some constant which may vary line to line.

(i) As in section 4.1, $\#\mathbf{X} \approx \lambda\mathrm{Vol}(L\mathcal{C})$ and for each $x \in \mathbf{X}$ we have $\Delta(G) \approx 2^d\lambda$.

(ii) For $x, y \in \mathbf{X}$, we have #triangles containing $(x,y) \approx \lambda\mathrm{Vol}(B(x;2r_d)\cap B(y;2r_d))$

(iii) By lemma 2.7, we have $\mathrm{Vol}(B(x;2r_d)\cap B(y;2r_d)) \lessapprox 2^d e^{-cr_d^2}$

(iv) By lemma 2.3, we have $r_d^2 \approx cd$.

(v) Using (iii) and (iv) in (ii), deduce #triangles containing $(x, y) \lessapprox \lambda 2^{d-cd}$

(vi) Take $\lambda = 2^{-d+cd}$ to obtain #triangles containing $(x, y) \ll 1$.

(vii) For such λ, G will be triangle free in expectation. Hence, by theorem 3.4

$$\alpha(G) \gtrapprox \frac{\log(2^d \lambda)}{2^d} \mathrm{Vol}(L\mathcal{C})$$

(viii) Deduce with equation (2)

$$\theta_{\mathcal{C}}(d; L) \gtrapprox cd2^{-d}$$

and take $L \to \infty$ to conclude.

Our primary lemma concerns the expected number of triangles in the geometric graph $G(\mathbf{X}, 2r_d)$. Before this, we'll need some technical lemmas.

Lemma 4.4. For $r > 0$, define

$$I(r) := \int_{B(\mathbf{0},r)} e^{-\|x\|^2} \mathrm{d}x \tag{27}$$

Then

$$I(r) = \frac{1}{2} \mathrm{Vol}(S^{d-1}) \gamma \left(\frac{d}{2}, r^2 \right) \tag{28}$$

where S^{d-1} denotes the boundary of the ball $B(\mathbf{0}, 1)$ and $\gamma(s, x) := \int_0^x t^{s-1} e^{-t} \mathrm{d}s$ denotes the lower incomplete gamma function.

Idea: Convert to hyperspherical coordinates.

Proof. The transformation $T : [0, r] \times S^{d-1} \to B(\mathbf{0}, r), \ (\rho, \omega) \mapsto \rho\omega$ is a well defined bijection. Further, it has Jacobian ρ^{d-1} so we compute

$$\int_{B(\mathbf{0},r)} e^{-\|x\|^2} \mathrm{d}x = \int_{S^{d-1}} \int_0^r e^{-\rho^2} \rho^{d-1} \mathrm{d}\rho \mathrm{d}\omega = \mathrm{Vol}(S^{d-1}) \int_0^r e^{-\rho^2} \rho^{d-1} \mathrm{d}\rho \tag{29}$$

For the remaining integral, apply the substitution $u = \rho^2$ to see that

$$\int_0^r e^{-\rho^2} \rho^{d-1} \mathrm{d}\rho = \frac{1}{2} \int_0^{r^2} e^{-u} u^{\frac{d}{2}-1} \mathrm{d}u = \frac{1}{2} \gamma \left(\frac{d}{2}, r^2 \right) \tag{30}$$

The desired equality follows immediately. $\square$

Lemma 4.5. Let $\gamma(s,x) := \int_0^x t^{s-1}e^{-t}\mathrm{d}s$ denote the lower incomplete gamma function. For all $d \in \mathbb{N}$ we have

$$\gamma\left(\frac{d}{2}, r_d^2\right) \leq \Gamma\left(\frac{d}{2}\right)\left(2\delta(d)e^{2\delta(d)+1}\right)^{\frac{d}{2}} \tag{31}$$

where

$$\delta(d) = \left(\frac{\pi(d+2)}{\exp\left(2-\frac{1}{3d+6}\right)}\right)^{\frac{1}{d}}\left(\frac{d+2}{2e\pi d}\right) \tag{32}$$

Idea: View the problem through a probabilistic lense.

Proof. Let Z be an $[0,\infty)$-valued random variable with probability density function

$$f_Z(z) = \frac{1}{\Gamma(d/2)}z^{\frac{d}{2}-1}e^{-z}$$

i.e. $Z \sim \mathrm{Gamma}(d/2, 1)$ where the left argument is the shape and the right argument is the rate. It is clear this random variable has cumulative density function

$$F_Z(z) = \frac{1}{\Gamma(d/2)}\gamma(d/2, z)$$

for $z > 0$. Hence

$$\gamma(d/2, r_d^2) = \Gamma(d/2)\mathbb{P}(Z \leq r_d^2) \tag{33}$$

Using lemma 2.4 and Markov's inequality, we bound for arbitrary $t \in (-\infty, 0)$

$$\mathbb{P}(Z \leq r_d^2) \leq \mathbb{P}(Z \leq \delta(d)d) = \mathbb{P}(e^{tZ} \geq e^{\delta(d)dt}) \leq e^{-\delta(d)dt}\mathbb{E}[e^{tZ}]$$

where $\delta(d)$ found by squaring $C(d)$ in equation (10) and given explicitly in equation (32). It is well known[39] that, for all $s \in (-\infty, 1)$, $\mathbb{E}[e^{sZ}] = (1-s)^{-\frac{d}{2}}$. Deduce

$$\mathbb{P}(Z \leq r_d^2) \leq e^{-\delta(d)dt}\left(1-t\right)^{-\frac{d}{2}}$$

We now seek to minimise the RHS over $t \in (-\infty, 0)$. To that end, note it is equivalent to minimise its logarithm

$$\delta(d)dt - \frac{d}{2}\log(1-t)$$

By differentiation we see this has argmin $t = 1 - \frac{1}{2\delta(d)}$. Note for $d \geq 1$ we have $t < 0$ so this is well defined. Hence we may bound, after some simplification,

$$\mathbb{P}(Z \leq r_d^2) \leq \left(2\delta(d)e^{1-2\delta(d)}\right)^{\frac{d}{2}}$$

and substitution into equation (33) establishes the lemma. ∎

[39]The moment generating function of the Gamma distribution is well known and easy to find.

Now we're ready to count the expected number of triangles in $G(\mathbf{X}, 2r_d)$. This lemma is of independent interest. We can use it show random geometric graphs inside a convex body sampled at certain densities $\lambda(d)$ will be triangle free in the limiting regime $d \to \infty$.

Lemma 4.6. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and $L > 0$. Let $\lambda > 0$ and sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$. Let $G := G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$. Let $\Gamma_3(G)$ denote the number of triangles in G. Then

$$\mathbb{E}\left[\Gamma_3(G)\right] \leq \lambda \mathrm{Vol}(L\mathcal{C}) \epsilon_2(d, \lambda)$$

where

$$\epsilon_2(d, \lambda) := \frac{\lambda^2 2^{2d} \pi^{\frac{d}{2}}}{6} \left(2\delta(d) e^{1-2\delta(d)}\right)^{\frac{d}{2}} \tag{34}$$

and $\delta(d)$ is given by equation (32).

Proof. For distinct points $x, y \in \mathbf{X}$ let $\Gamma_3(G, \{x, y\})$ denote the number of triangles in G containing x and y. It is clear that

$$\mathbb{E}\left[\Gamma_3(G)\right] = \frac{1}{6} \cdot \mathbb{E}\left[\sum_{X,Y \in \mathbf{X}}^{\neq} \Gamma_3(G, \{x, y\})\right]$$

where $\sum_{X_1,\ldots,X_k \in \mathbf{X}}^{\neq}$ denotes summation over k-tuples of distinct points in $\mathbf{X}$. Applying Mecke's formula (theorem 2.13) we see that

$$\mathbb{E}\left[\Gamma_3(G)\right] = \frac{\lambda^2}{6} \int_{L\mathcal{C}} \int_{L\mathcal{C}} \mathbb{E}\left[\Gamma_3(G_{x,y}, \{x, y\})\right] \mathrm{d}x \mathrm{d}y$$

where $G_{x,y} := G(\mathbf{X} \cup \{x\} \cup \{y\}, 2r_d)$ denotes the geometric graph G with vertices $\mathbf{X} \cup \{x, y\}$ and radius $2r_d$. As $x \sim y \Rightarrow x \in B(y; 2r_d)$, the RHS reduces to

$$\frac{\lambda^2}{6} \int_{L\mathcal{C}} \int_{B(y;2r_d)} \mathbb{E}\left[\Gamma_3(G_{x,y}, \{x, y\})\right] \mathrm{d}x \mathrm{d}y$$

Writing

$$\Gamma_3(G_{x,y}, \{x, y\}) = \#\{z \in \mathbf{X} : \|z - x\|, \|z - y\| \leq 2r_d\}$$

it is clear that $\Gamma_3(G_{x,y}, \{x, y\}) = \#\mathbf{X} \cap B(x; 2r_d) \cap B(y; 2r_d)$. Hence,

$$\Gamma_3(G_{x,y}, \{x, y\}) \sim \mathrm{Po}(\lambda \mathrm{Vol}(B(x; 2r_d) \cap B(y; 2r_d) \cap L\mathcal{C}))$$

and we deduce $\Gamma_3(G_{x,y}, \{x, y\})$ is dominated by a $\mathrm{Po}(\lambda \mathrm{Vol}(B(x; 2r_d) \cap B(y; 2r_d)))$ random variable. Let $Z_{x,y} \sim \mathrm{Po}(\lambda 2^d \exp(\|x - y\|^2/4))$ independent of $\mathbf{X}$. With lemma 2.7, we can dominate $\Gamma_3(G_{x,y}, \{x, y\})$ with $Z_{x,y}$. Hence,

$$\mathbb{E}\left[\Gamma_3(G)\right] \leq \frac{\lambda^2}{6} \int_{L\mathcal{C}} \int_{B(y;2r_d)} \mathbb{E}[Z_{x,y}] \mathrm{d}x \mathrm{d}y = \frac{2^d \lambda^3}{6} \int_{L\mathcal{C}} \int_{B(y;2r_d)} e^{\frac{\|x-y\|^2}{4}} \mathrm{d}x \mathrm{d}y$$

Apply the substitution $z = \frac{x-y}{2}$ to deduce the equality

$$\int_{L\mathcal{C}} \int_{B(y;2r_d)} e^{\frac{\|x-y\|^2}{4}} \mathrm{d}x\mathrm{d}y = \mathrm{Vol}(L\mathcal{C})2^d \int_{B(\mathbf{0};r_d)} e^{\|z\|^2} \mathrm{d}z$$

By lemma 4.4 we see the remaining integral evaluates to $\frac{1}{2}\mathrm{Vol}(s^{d-1})\gamma(\frac{d}{2}, r_d^2)$, hence

$$\int_{L\mathcal{C}} \int_{B(y;2r_d)} e^{\frac{\|x-y\|^2}{4}} \mathrm{d}x\mathrm{d}y = \mathrm{Vol}(L\mathcal{C})\mathrm{Vol}(S^{d-1})2^{d-1}\gamma\left(\frac{d}{2}, r_d^2\right)$$

and all together we may bound

$$\mathbb{E}\left[\Gamma_3(G)\right] \leq \frac{\lambda^3 2^{2d}}{12}\mathrm{Vol}(L\mathcal{C})\mathrm{Vol}(S^{d-1})\gamma\left(\frac{d}{2}, r_d^2\right)$$

Using lemma 2.3 and lemma 4.5, deduce

$$\mathbb{E}\left[\Gamma_3(G)\right] \leq \lambda\mathrm{Vol}(L\mathcal{C}) \underbrace{\left(\frac{\lambda^2 2^{2d}\pi^{\frac{d}{2}}}{6}\left(2\delta(d)e^{1-2\delta(d)}\right)^{\frac{d}{2}}\right)}_{=\epsilon_2(d,\lambda)}$$

where $\delta(d)$ is given in equation (32). ☐

Remark 4.7. More generally, bounds on the expected number of copies of a graph H in a random geometric graph in the hypercube $[0,1]^d$ are given in [27].

We're now ready to prove $\theta(d) = \Omega(2^{-d}d)$.

Theorem 4.8. Let $d \geq 100$. Then

$$\theta(d) \geq \kappa(d)d2^{-d}$$

where

$$\kappa(d) := \left(\frac{\log(2)c(d)\left(1 - \exp\left(2^{\frac{1}{3}c(d)d}\right) - \frac{1}{6}\exp(-2\log(2)d^{0.1})\right)}{\left(1 + 2^{-\frac{1}{3}c(d)d}\right) + 2^{-c(d)d}}\right)$$

$$c(d) := \frac{2\delta(d) - 1 - \log(\delta(d)) - \log(\pi) - \log(2)}{4\log(2)} - d^{-0.9}$$

and $\delta(d)$ is given by equation (32). In particular,

$$\theta(d) \geq (1 - o(1))\tilde{c}d2^{-d}$$

where $o(1) \xrightarrow{d\to\infty} 0$ and $\tilde{c} = \frac{1}{4\pi e} \approx 0.0293$.

For typographical purposes, define the exponentiation map $\ell_a : \mathbb{R} \to \mathbb{R}, x \mapsto a^x$.

Proof. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and let $L > 0$. Let $\lambda = \ell_2(-d + c(d)d)$. Note that $c(d) > 0$ for all $d \geq 100$. Sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$ and let $G := G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$. Let $\Delta = \left(1 + \ell_2\left(-\frac{1}{3}c(d)d\right)\right) 2^d\lambda + 1$ and

$$B_1 := \{x \in \mathbf{X} : d_G(x) > \Delta(d, \lambda)\}$$

be the vertices of G with degree greater than Δ. Let

$$B_2 := \{x \in \mathbf{X} : \exists y, z \in \mathbf{X} \text{ s.t. } \forall i, j \in \{x, y, z\} \text{ with } i \neq j, i \sim j \text{ and } \mathrm{LMP}(\{x, y, z\}) = x\}$$

where $\mathrm{LMP}(A)$ denotes the furthest left point for a finite set $A \subset \mathbb{R}^d$ in the standard[40] lexicographical ordering of $\mathbb{R}^d$. This is the collection of leftmost vertices of triangles in G. With lemma 4.1 and lemma 4.6, bound

$$\mathbb{E}[\mathbf{X} \setminus (B_1 \cup B_2)] \geq \mathbb{E}[\mathbf{X}] - \mathbb{E}[B_1] - \mathbb{E}[B_2] \geq \lambda \mathrm{Vol}(L\mathcal{C})(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_2(d, \lambda))$$

where $\epsilon_1(d, \lambda, \Delta)$ is given in equation (26) and $\epsilon_2(d, \lambda)$ is given in equation (34). By lemma 2.16 we have the existence of $\mathbf{Y} \subset \mathbb{R}^d$ with $\#\mathbf{Y} \geq \lambda \mathrm{Vol}(L\mathcal{C})(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_2(d, \lambda))$, $\Delta(H) \leq \Delta$ and H triangle-free[41], where $H := H(\mathbf{Y}, 2r_d)$ is the geometric graph with vertex set $\mathbf{Y}$ and radius $2r_d$. By theorem 3.4 we have

$$\alpha(H) \geq \frac{\log \Delta}{\Delta} \lambda \mathrm{Vol}(L\mathcal{C}) \left(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_2(d, \lambda)\right)$$

Using equation (2), deduce

$$\theta_{\mathcal{C}}(d; L) \geq \frac{\log \Delta}{\Delta} \lambda \left(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_2(d, \lambda)\right) \tag{35}$$

As $\ell_a(x) = \exp(\log(a)x)$, compute

$$\begin{aligned}
\epsilon_2(d, \lambda) &= \frac{1}{6} \ell_2 \left(\frac{d}{2} + 2c(d)d\right) \ell_\pi \left(\frac{d}{2}\right) \ell_{\delta(d)} \left(\frac{d}{2}\right) \exp\left(\frac{d}{2} - d\delta(d)\right) \\
&= \frac{1}{6} \exp\left(\frac{1}{2}\log 2 + 2c(d)\log 2 + \frac{1}{2}\log \pi + \frac{1}{2}\log \delta(d) + \frac{1}{2} - \delta(d)\right)^d \\
&= \frac{1}{6} \exp(-2\log(2)d^{0.1}) = o(1)
\end{aligned} \tag{A}$$

[40]Let $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. We have x to the left of y if $x_1 < y_1$ or if $x_1 = y_1$ and $x_2 < y_2$ or $\cdots$ or $x_1 = y_1$ and $\cdots$ and $x_{d-1} = y_{d-1}$ and $x_d < y_d$.

[41]We have deleted a vertex from each triangle by removing $B_2 \subset \mathbf{X}$, hence the clique number of H is bounded above by 2 and we have the triangle free condition.

and

$$\epsilon_1(d,\lambda,\Delta) = \exp\left(\ell_2\left(\frac{1}{3}c(d)d\right)\right) = o(1) \tag{B}$$

where $o(1) \xrightarrow{d\to\infty} 0$. Deduce

$$\epsilon_1(d,\lambda,\Delta) + \epsilon_2(d,\lambda) = o(1) \tag{C}$$

Compute

$$\begin{aligned}
\lambda\frac{\log\Delta}{\Delta} &= \frac{\ell_2(-d+c(d)d)\log\left(\left(1+\ell_2\left(-\frac{1}{3}c(d)d\right)\right)\ell_2(c(d)d)+1\right)}{\left(1+\ell_2\left(-\frac{1}{3}c(d)d\right)\right)\ell_2(c(d)d)+1} \\
&\geq \left(\frac{\log\left(1+\ell_2\left(-\frac{1}{3}c(d)d\right)\right)+\log(2)c(d)d}{\left(1+\ell_2\left(-\frac{1}{3}c(d)d\right)\right)+\ell_2(-c(d)d)}\right)2^{-d} \\
&\geq \left(\frac{\log(2)c(d)}{\left(1+\ell_2\left(-\frac{1}{3}c(d)d\right)\right)+\ell_2(-c(d)d)}\right)d2^{-d} \qquad \text{(D)}
\end{aligned}$$

Substituting equations (A), (B) and (D) into equation (35), deduce

$$\theta_{\mathcal{C}}(d;L) \geq \underbrace{\left(\frac{\log(2)c(d)\left(1-\exp\left(\ell_2\left(\frac{1}{3}c(d)d\right)\right)-\frac{1}{6}\exp(-2\log(2)d^{0.1})\right)}{\left(1+\ell_2\left(-\frac{1}{3}c(d)d\right)\right)+\ell_2(-c(d)d)}\right)}_{=\kappa(d)}d2^{-d}$$

Letting $L\to\infty$ establishes the first part of theorem. For the second part, substitute equations (C) and (D) into equation (35) to see

$$\theta_{\mathcal{C}}(d;L) \geq \underbrace{\left(\frac{\log(2)c(d)}{\left(1+\ell_2\left(-\frac{1}{3}c(d)d\right)\right)+\ell_2(-c(d)d)}\right)}_{=(1-o(1))\log(2)c(d)}(1-o(1))d2^{-d}$$

where $o(1) \xrightarrow{d\to\infty} 0$. By taking $L\to\infty$ followed by $d\to\infty$, deduce

$$\theta(d) \geq \log(2)\left(\lim_{d\to\infty} c(d)\right)d2^{-d}(1-o(1))$$

It is easy to spot $\lim_{d\to\infty} c(d) = \frac{1}{4\log(2)\pi e}$, completing the proof. □

Remark 4.9. Our bound is of an identical form to Krivelevich and Vardy's [18], but with a marginally stronger constant. Further, we obtain bounds for fixed dimension. It is likely this constant can be sharpened further by, e.g. using a tighter covolume bound, but with theorem 4.11 this task becomes uninteresting.

4.3 $\theta(d) = \Omega(2^{-d} d \log d)$

We again begin with a sketch argument. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and $L > 0$. Let $\lambda > 0$, sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$ and let $G = G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$.

(i) As in section 4.1, $\#\mathbf{X} \approx \lambda \mathrm{Vol}(L\mathcal{C})$ and $\Delta(G) \approx 2^d \lambda$.

(ii) Delete $x \in \mathbf{X}$ if $\exists y \in \mathbf{X} \setminus \{x\}$ such that $\|x - y\| \leq \log d$. There are approximately $\lambda \mathrm{Vol}(B_{\mathbf{0}}(\log d)) = \lambda r_d^{-d} (\log d)^d \approx \left(\frac{\log d}{c\sqrt{d}}\right)^d$ such points by lemma 2.5. Hence, for there to be $o(\#\mathbf{X})$ such points we take at most $\lambda = (c\sqrt{d}/\log d)^d$.

(iii) After deleting these points, we have

$$d_{G(\mathbf{X})}(x, y) \approx \lambda \mathrm{Vol}(B(x; 2r_d) \cap B(y; 2r_d)) \lessapprox \exp\left(-\frac{(\log d)^2}{4}\right) \Delta$$

by lemma 2.7.

(iv) Deduce $\Delta_{\mathrm{co}}(G) \lessapprox (2 \log \Delta(G))^{-7} \Delta$ so that we may use theorem 3.7 to bound

$$\alpha(G) \gtrapprox (1 - o(1)) \frac{\log(\Delta(G)) \#\mathbf{X}}{\Delta(G)} \approx (1 - o(1)) 2^{-d-1} d \log d \mathrm{Vol}(L\mathcal{C})$$

From here $\theta(d) \gtrapprox (1 - o(1)) 2^{-d-1} d \log d$ is clear.

Here we prove there aren't too many points that have codegree too large for an application of theorem 3.7. We split on the cases of $\|x - y\| \leq \log d$, noting that later we will choose λ so that there aren't too many points with $\|x - y\| \leq \log d$ true.

Lemma 4.10. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and $L > 0$. Let $\lambda > 0$ and sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$. Let $G = G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$. Let $\Delta, \eta > 0$. Then

$$\mathbb{E}\left[\{x \in \mathbf{X} : \exists\, y \in \mathbf{X} \text{ s.t. } d_G(x, y) \geq \eta\Delta\}\right] \leq \lambda \mathrm{Vol}(L\mathcal{C}) \epsilon_3(d, \lambda, \Delta, \eta)$$

where

$$\epsilon_3(d, \lambda, \Delta, \eta) := \frac{1}{\lambda}\left(\mathrm{Vol}(B(x; \log d)(1 - e^{-\eta\Delta}) + 4^d e^{-\eta\Delta})\right) \tag{36}$$

Proof. For $x, y \in \mathbb{R}^d$, define $Z_{x,y} := \#\mathbf{X} \cap B(x; 2r_d) \cap B(y; 2r_d)$. We have $Z_{x,y} \sim \mathrm{Po}(\lambda \mathrm{Vol}(B(x; 2r_d) \cap B(y; 2r_d) \cap L\mathcal{C}))$ which is clearly dominated by an independent $\mathrm{Po}(\lambda \mathrm{Vol}(B(x; 2r_d), B(y; 2r_d)))$ random variable. By lemma 2.7 this is itself dominated by an independent $\mathrm{Po}(\lambda 2^d \exp(-\|x-y\|^2/4)) =: Z'_{x,y}$ random variable. By Mecke's formula (theorem 2.13), we deduce

$$\mathbb{E}\left[\{x \in \mathbf{X} : \exists y \in \mathbf{X} \text{ s.t. } Z_{x,y} > \eta\Delta\}\right] \leq \lambda \int_{L\mathcal{C}} \mathbb{P}(\exists y \in \mathbf{X} \text{ s.t. } Z'_{x,y} \geq \eta\Delta)\mathrm{d}x \tag{37}$$

Splitting on $B(x;\log d)$ and applying Markov's inequality twice, we see

$$\begin{aligned}
&\mathbb{P}(\exists y \in \mathbf{X} \text{ s.t. } Z'_{x,y} \geq \eta\Delta)\\
&= \mathbb{P}(\exists y \in B(x;\log d) \text{ s.t. } Z'_{x,y} \geq \eta\Delta) + \mathbb{P}(\exists y \in \mathbf{X} \setminus B(x;\log d) \text{ s.t. } Z'_{x,y} \geq \eta\Delta)\\
&\leq \mathbb{E}[\#\{y \in B(x;\log d) : Z'_{x,y} \geq \eta\Delta\}] + \mathbb{E}[\#\{y \in \mathbf{X} \setminus B(x;\log d) : Z'_{x,y} \geq \eta\Delta\}]\\
&\leq \mathbb{E}[\#\mathbf{X} \cap B(x;\log d)] + \mathbb{E}[\#\{y \in \mathbf{X} \setminus B(x;\log d) : Z'_{x,y} \geq \eta\Delta\}]\\
&= \lambda \mathrm{Vol}(B(x;\log d)) + \mathbb{E}[\#\{y \in \mathbf{X} \setminus B(x;\log d) : Z'_{x,y} \geq \eta\Delta\}] \qquad \text{(A)}
\end{aligned}$$

where the final equality is due to lemma 2.3. For the second term, we apply Mecke's formula (theorem 2.13) again to bound[42]

$$\mathbb{E}[\#\{y \in \mathbf{X} \setminus B(x;\log d) : Z'_{x,y} \geq \eta\Delta\}] \leq \lambda \int_{B(x;4r_d)\setminus B(x;\log d)} \mathbb{P}(Z'_{x,y} \geq \eta\Delta - 1)\mathrm{d}y$$

It is clear that the right hand side is bounded from above by

$$\mathrm{Vol}(B(\mathbf{0};4r_d) \setminus B(\mathbf{0};\log d)) \sup_{y \in B(\mathbf{0};4r_d)\setminus B(\mathbf{0};\log \mathbf{d})} \mathbb{P}(Z'_{\mathbf{0},y} \geq \eta\Delta - 1)$$

Let $Z \sim \mathrm{Po}(\lambda 2^d)$ indepedently of the $Z_{\mathbf{0},y}$. The supremum is attained when $y = \mathbf{0}$, hence

$$\lambda \int_{B(x;4r_d)\setminus B(x;\log d)} \mathbb{P}(Z'_{x,y} \geq \eta\Delta - 1)\mathrm{d}y \leq \lambda \mathrm{Vol}(B(\mathbf{0};4r_d) \setminus B(\mathbf{0};\log d))\mathbb{P}(Z \leq \eta\Delta - 1) \quad \text{(B)}$$

Applying lemma 2.10, see

$$\mathbb{P}(Z \leq \eta\Delta - 1) \leq \mathbb{P}\left(Z - 2^d\lambda \leq 2^d\lambda \cdot \frac{\eta\Delta}{2^d\lambda}\right) \leq \exp(-\eta\Delta) \qquad \text{(C)}$$

Using (A), (B) and (C) in (37), deduce

$$\mathbb{E}\left[\{x \in \mathbf{X} : \exists y \in \mathbf{X} \text{ s.t. } Z_{x,y} \geq \eta\Delta\}\right] \leq \lambda^2 \mathrm{Vol}(L\mathcal{C}) \underbrace{\left(\mathrm{Vol}(B(x;\log d)(1 - e^{-\eta\Delta}) + 4^d e^{-\eta\Delta}\right)}_{=\epsilon_3(d,\lambda,\Delta,\eta)/\lambda}$$

which completes the proof. $\square$

[42]Mecke's formula gives the expression but with the integral over $L\mathcal{C} \setminus B(x;\log d) \subset \mathbb{R}^d \setminus B(x;\log d)$, but our probability is 0 outside of $B(x;4r_d)$ so our bound follows.

We finish with our key theorem. We work analogously to the proof of theorem 4.8. Again, most of the difficulty comes from finding a clean, explicit, bound for sufficiently large d.

Theorem 4.11. Let $d \geq 1777$ and $\lambda = 2^{-3d} d^{d/2} (\log d)^{-d}$. Then

$$\theta(d) \geq \kappa_2(d) 2^{-d-1} d \log(d)$$

where

$$\kappa_2(d) := \left(1 - \frac{1 + 2^{-\frac{d}{3}} \lambda^{-\frac{1}{3}}}{2^d \lambda}\right) \left(1 - \frac{40 \log\log\lambda}{\log\lambda}\right) \left(1 - \exp\left(-2^{\frac{d}{3}} \lambda^{\frac{1}{3}}\right) - d^{-\frac{d}{2}} 2^{3d}\right)$$

In particular,

$$\theta(d) \geq (1 - o(1)) 2^{-d-1} d \log d$$

Proof. Let $\mathcal{C} \subset \mathbb{R}^d$ be a convex body and let $L > 0$ satisfy $\mathrm{Vol}(L\mathcal{C}) \geq 1$. Sample $\mathbf{X} \sim \mathrm{PPP}_{L\mathcal{C}}(\lambda)$ and let $G = G(\mathbf{X}, 2r_d)$ be the geometric graph with vertex set $\mathbf{X}$ and radius $2r_d$. Let $\Delta = 2^d \lambda (1 + 2^{-d/3} \lambda^{-1/3})$ and $\eta = \frac{\exp(-(\log d)^2/8)}{1 + 2^{-d/3} \lambda^{-1/3}}$. Let $B_1 := \{x \in \mathbf{X} : d_G(x) > \Delta\}$ and $B_2 := \{x \in \mathbf{X} : \exists y \text{ s.t. } d_G(x, y) > \eta\Delta\}$. Applying lemma 4.1 and lemma 4.10, bound

$$\mathbb{E}[\mathbf{X} \setminus (B_1 \cup B_2)] \geq \mathbb{E}[\mathbf{X}] - \mathbb{E}[B_1] - \mathbb{E}[B_2] \geq \lambda \mathrm{Vol}(L\mathcal{C}) \left(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_3(d, \lambda, \Delta, \eta)\right)$$

where $\epsilon_1(d, \lambda, \Delta)$ is given in equation (26) and $\epsilon_3(d, \lambda, \Delta, \eta)$ in (36). Hence, by lemma 2.16, $\exists \mathbf{Y} \subset \mathbb{R}^d$ with the geometric graph $H := G(\mathbf{Y}, 2r_d)$ satisfying

$$\# V(H) \geq \lambda \mathrm{Vol}(L\mathcal{C})(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_3(d, \lambda, \Delta, \eta)), \ \Delta(H) \leq \Delta \text{ and } \Delta_{\mathrm{co}}(H) \leq \eta\Delta$$

where ϵ_1 is given in (26) and ϵ_3 in (36). First observe that, with room to spare[43],

$$\eta \leq (2 \log \Delta)^{-7} \quad \text{and} \quad \Delta \geq 160^{160}$$

Hence, H satisfies the requirements of theorem 3.7 and we have

$$\alpha(H) \geq \frac{\log \Delta}{\Delta} \left(1 - \frac{40 \log\log\Delta}{\log\Delta}\right) \lambda \mathrm{Vol}(L\mathcal{C}) \left(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_3(d, \lambda, \Delta, \eta)\right)$$

Use (2) and take $L \to \infty$ to deduce

$$\theta(d) \geq \frac{\lambda \log \Delta}{\Delta} \left(1 - \frac{40 \log\log\Delta}{\log\Delta}\right) \left(1 - \epsilon_1(d, \lambda, \Delta) - \epsilon_3(d, \lambda, \Delta, \eta)\right) \tag{38}$$

We bound

$$\epsilon_1(d, \lambda, \Delta) = \exp\left(-\frac{1}{2^d \lambda} (\Delta - 1 - 2^d \lambda)^2\right) \leq \exp\left(-2^{d/3} \lambda^{1/3}\right) = o(1) \tag{A}$$

[43] For the latter inequality, we use calculus to find the minima and then a numerical sim to find where it breaks this threshold. I find this to be $d = 1777$.

By lemmas 2.3 and 2.5, we also bound

$$\begin{aligned}
\epsilon_3(d,\lambda,\Delta,\eta) &= \frac{1}{\lambda}\left(\mathrm{Vol}(B(\mathbf{0},\log d))(1-e^{-\eta\Delta}) + 4^d e^{-\eta\Delta}\right) \\
&= d^{-\frac{d}{2}}\pi^{\frac{d}{2}}2^{3d}(\Gamma(d/2+1))^{-1}(1-e^{-\eta\Delta}) + d^{-d/2}2^{5d}(\log d)^{-d}e^{-\eta\Delta} \\
&\le d^{-d/2}2^{3d}\left(r_d^{-d} + 2^{2d}(\log d)^{-d}e^{-\eta\Delta}\right) \\
&\le d^{-d/2}2^{3d}\left(\frac{\sqrt{2e^3}}{(d+2)^{(d+1)/2}} + 2^{2d}(\log d)^{-d}e^{-\eta\Delta}\right) \\
&\le d^{-d/2}2^{3d} = o(1) \qquad \text{(B)}
\end{aligned}$$

For $x,\delta>0$, we have[44] $\frac{1}{x+\delta} \ge \frac{1}{x}\left(1-\frac{\delta}{x}\right)$. Hence,

$$\begin{aligned}
\frac{\log\Delta}{\Delta} &\ge \frac{d\log d - 2d\log(4\log d)}{2^{d+1}\lambda}\left(1-\frac{1+2^{-d/3}\lambda^{-1/3}}{2^d\lambda}\right) \\
&\ge \frac{d\log d}{2^{d+1}\lambda}\left(1-\frac{1+2^{-d/3}\lambda^{-1/3}}{2^d\lambda}\right) \\
&= \frac{d\log d}{2^{d+1}\lambda}(1-o(1)) \qquad \text{(C)}
\end{aligned}$$

As $\log(x+y) \le \log 2 + \max\{\log x, \log y\}$,

$$\begin{aligned}
\frac{\log\log\Delta}{\log\Delta} &\le \frac{\log 2 + \log\log(2^d\lambda)}{\log\lambda} \\
&\le \frac{\log\log(2^d\lambda)}{\log\lambda}\left(1+\frac{1}{(\log\lambda)(\log\log(2^d\lambda))}\right) \\
&\le \frac{\log 2 + \log\log\lambda}{\log\lambda}\left(1+\frac{1}{(\log\lambda)(\log\log(2^d\lambda))}\right) \\
&\le \frac{\log\log\lambda}{\log\lambda}\left(1+\frac{1}{(\log\lambda)(\log\log(2^d\lambda))}\right)\left(1+\frac{1}{(\log\lambda)(\log\log\lambda)}\right) \\
&\le \frac{\log\log\lambda}{\log\lambda}\left(1+\frac{1}{(\log\log\lambda)^2}\right)^2 = o(1) \qquad \text{(D)}
\end{aligned}$$

Thus, using (A)-(D) in (38), deduce

$$\theta(d) \ge \underbrace{\left(1-\frac{1+2^{-\frac{d}{3}}\lambda^{-\frac{1}{3}}}{2^d\lambda}\right)\left(1-\frac{40\log\log\lambda}{\log\lambda}\right)\left(1-\exp\left(-2^{\frac{d}{3}}\lambda^{\frac{1}{3}}\right)-d^{-\frac{d}{2}}2^{3d}\right)}_{=\kappa_2(d)}\frac{d\log d}{2^{d+1}}$$

[44]This is a consequence of the tangent line bound for convex functions.

and, in particular,

$$\theta(d) \geq (1 - o(1))2^{-d-1} d \log d$$

which completes the proof. $\square$

Remark 4.12. We could weaken the requirement on d by letting L be so that $\mathrm{Vol}(L\mathcal{C}) = c > 1$, our choice of $c = 1$ was purely for simplicity. In [3] $d \geq 1000$ is taken, which follows from taking $L = 1$ and $\mathcal{C} = B(\mathbf{0}, 1)$. We could've sharpened our bounds to get $\kappa_2(d) \to 0$ faster but, for the purpose of exposition, our simpler $\kappa_2(d)$ makes for a better choice.

5 Contributions, Remarks & Future Work

Contributions

Theorems 2.14 and 4.8 are novel, using similar ideas[45] to those in [3]. Lemma 2.8 is novel as is the proof of lemma 2.9. Our bounds for r_d given in lemma 2.5 are novel, but similar results almost certainly exist elsewhere. We extend Theorems 1.1 and 1.3 in [3] by providing bounds for fixed d and Δ respectively. Our sketch arguments using the concentration of Poisson random variables about their mean (i.e. rate) are novel, though were likely discussed by the authors of [3]. Our proof of lemma 2.10 is novel, though the approach is classical.

Future Work

Originally, it was planned to also provide sphere packing bounds in convex bodies $\mathcal{C} \subset \mathbb{R}^d$. When attempting this, one seeks a bound on the relative desnity $\mathcal{D}(\mathcal{P}, \mathbb{R}^d \setminus \mathcal{C})$ in order to obtain an explicit statement in fixed dimensions. It is clear this volume is $o(1)$ where $o(1) \to 0$ as $d \to \infty$ for sufficiently large $\mathcal{C}$, so obtaining asymptotic results is easy.

For fixed dimensions, the problem is not so easy. We have the "dual"[46] problem of finding the "worst" sphere packing, in the sense of having the most volume outside of $\mathcal{C}$. It seems likely[47] that this is when we have a maximal number of d-balls on $\partial\mathcal{C}$. If we can prove this is the case, then there we can give a simple expression for $\mathcal{D}(\mathcal{P}, \mathbb{R}^d \setminus \mathcal{C})$ in terms of $\mathrm{Vol}_{d-1}(\partial\mathcal{C})$ by Dirichlet's principle[48]. If a proof of this is found, an addendum shall be released including bounds for the quantity $\theta_{\mathcal{C}}(d; L)$.

It is also of interest to consider stochastic algorithms for sphere packings using the Cox process machinery. By considering a sequence of realisations of a Cox process, I predict we can obtain an algorithm that obtains a sphere packing of density $(1 - o(1))2^{-d-1} d \log d$ with high probability. If this is possible, this work will be given in a future paper.

I plan to also attempt to use this Cox machinery to compare lattice packings and amorphous packings. Fix a lattice $\mathcal{L}$ and consider a Cox process directed by the random field $f(x) = \min_{a \in \mathcal{L}} \|x - a\|$. Intuitively this process biases our random packing towards the lattice $\mathcal{L}$. If it can be shown that some realisation of this process has greater density than the packing given by placing d-balls centred at the lattice points $\mathcal{L}$, we can exhibit the existence of a phase transition. I predict there will be significant technical difficulty and

[45] That were found independently.

[46] This is not a dual problem in the original sense of the word, though it is in spirit.

[47] It certainly should be the case when $\mathcal{C}$ has a flat boundary, e.g. $\mathcal{C} = [-1, 1]^d$, but it seems like it could be false for $\mathcal{C}$ with an interesting boundary.

[48] This is often called the pigeonhole principle.

further, it is likely that this approach leads to a dead-end.

Finally, I believe an argument similar to Krivelevich and Vardy's [18] can be used to obtain a deterministic procedure for finding packings of density $(1 - o(1))2^{-d-1}d \log d$. This will be explored in the months that follow.

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